APPLICATIONS IN COMMUTATIVE ALGEBRA OF THE MOORE COMPLEX OF A SIMPLICIAL ALGEBRA

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Applications in Commutative Algebra of the Moore Complex of a Simplicial Algebra

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Dedicated

to

the memory of my father

Declaration

The work of this thesis has been carried out by the candidate and contains the results of his own investigations. The work has not been already accepted in substance for any degree, and is not being concurrently submitted in candidature for any degree. All sources of information have been acknowledged in the text.

Director of Studies　　　　　　　　　　　　Candidate

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Contents

Summary

The first chapter presents and develops some of the basics of simplicial algebra theory and the elementary theory of crossed modules of commutative algebras. It contains a 'step-by-step' construction of a free simplicial algebra with given homotopy modules. Some results regarding this construction are extented.

Chapter Two generalises the 'higher order Peiffer elements' for commutative algebras to dimension 2, 3 and 4 and obtains partial results in higher dimensions.

Chapter Three gives a functor from the category of simplicial algebras to that of crossed complexes. A direct proof for simplicial algebras is given without needing understanding of the hypercrossed complex structure used by Carrasco and Cegarra. There is also a section recalling the particular case of the 'step-by-step' construction and giving many of the basic technical results that relate various structures. Using these data and the higher order Peiffer elements, we can form a free crossed resolution of a commutative algebra.

Chapter Four and Five mainly study 2-crossed modules, crossed squares and the freeness case of those structures. Applying the higher order Peiffer elements, we explore the relations between the structures mentioned above. This information and the 'step-by-step' construction with its k-skeleton are applied to describe algebraic models of the n-type of the k-skeleton of a free simplicial algebra.

The last two chapters also provide a functor from simplicial algebras to crossed n-cubes and use all these data to analyse the connections between free 2-crossed modules and free crossed squares.

Chapter 0

Introduction

The original motivation for this research was to see what parts of the group theoretic case of crossed homotopical algebra generalised to the context of commutative algebras and to see how existing parts of commutative algebra might interact with the analogue. The hope was for a clarification of the group theoretic situation as well as perhaps introducing 'new' tools into commutative algebra. The existing theory of crossed modules and crossed complexes within commutative algebras (in [36]) led to the realisation that the Koszul complex was linked with the construction of a free crossed module. In the group theoretic setting the interpretation of the levels immediately beyond that of a 'presentation' has only just started (about five years ago) and it is still very unclear what this tells one. This is equally true in commutative algebra. The idea was that André's step-by-step construction of simplicial resolution gave a good means of revealing some of the problems and questions hidden in these first few levels.

R.Brown and J-L.Loday [10] have noted that if the second dimension G_2 of a simplicial group G is generated by degenerate elements, that is elements coming from lower dimensions, then the image of the second term NG_2 of the Moore complex (NG, ∂) of G by the differential, ∂, is

$$[\mathrm{Ker} d_0, \mathrm{Ker} d_1]$$

where the square brackets denote the commutator subgroup. An easy argument then shows that this subgroup of NG_1 is generated by elements of the form $(yxy^{-1})(s_0d_1(y)x^{-1}s_0d_1(y)^{-1})$ and that it is thus exactly the Peiffer subgroup of NG_1, the vanishing of which is equivalent to $\partial : NG_1 \to NG_0$ being a crossed module.

It is clear that one should be able to develop an analogous result for other algebraic structures and in the case of commutative algebras, it is not difficult to see that if **E** is a simplicial algebra in which the subalgebra, E_2, is generated by the degenerate elements then the corresponding image is the ideal $\mathrm{Ker}d_0\mathrm{Ker}d_1$ in NE_1 and that it is generated by the elements $x(s_0d_1y-y)$ (see section 2.4.1) this gives the analogous Peiffer *ideal* for the theory of crossed modules of algebras. Given the importance of the vanishing of these elements in the construction of the cotangent complex of Lichtenbaum and Schlessinger, [31], and the simplicial version of the cotangent complex of Quillen [39], André [1] and Illusie [26], it is natural to hope for higher order analogues of this result and for an analysis and interpretation of the structure of the resulting elements in $NE_n,\ n \geq 2$. In this thesis, the analysis of these higher elements has been extended to dimension four and partial results obtained in higher dimensions.

M.André [1] and D.Quillen [39] developed the theories of homotopical algebra and that of simplicial algebras. They constructed ways of building simplicial resolutions of algebras, called a 'step-by-step' construction, and defined a homology and cohomology of commutative algebras, which can be 'computed' by means of this resolution. The 'step-by-step' construction of a free simplicial algebra is fundamental to the subject matter of this thesis.

The purpose of this thesis is to analyse the 'Higher Order Peiffer Elements' and to search for potential applications arising in the 'step–by-step' construction of free simplicial algebras. The study of the Peiffer elements shows that there are relations between the commutative algebra analogues of 2-crossed modules and of crossed squares. Applications of the free simplicial algebra give the freeness feature for those structures in terms of the Peiffer elements. In addition, the 'step-by-step' construction with its k-skeleton is applied to define algebraic models of n-types of the k-skeleton of a free simplicial algebra.

0.1 Structure of Thesis

We commence Chapter 1 by giving some general results on simplicial algebras and homotopical algebra. The construction of simplicial resolutions is studied in this chapter. This material is not easy to read in the literature and an attempt has been made to give a clear exposition. We give an explicit explanation of that together with the basic geometric pictures and also note the result which says: if **A** is a simplicial algebra, then there exists a free simplicial algebra **E** and an epimorphism

$$\mathbf{E} \longrightarrow \mathbf{A}$$

which induces isomorphisms on all homotopy modules.

In addition we collect together the elementary theory of crossed modules of commutative algebras. We will often use from [36] the link between free crossed modules and Koszul complexes (Proposition 1.5.2).

In chapter 2 of this thesis, we generalise the Peiffer elements for commutative algebras to dimensions 2, 3 and 4 and get partial results in higher dimensions. The methods we use are based on ideas of Conduché, [14], and techniques developed by Carrasco and Cegarra, [13]. In detail, this gives the following:

Let **E** be a simplicial commutative algebra with Moore complex **NE** and for $n > 1$, let D_n be the ideal generated by the degenerate elements in dimension n. If $E_n = D_n$, then

$$\partial_n(NE_n) = \partial_n(I_n) \quad \text{for all } n > 1$$

where I_n is an ideal in E_n generated by a fairly small set of elements which can be explicitly given.

If $n = 2, 3$ or 4, then the image by ∂_n of the Moore complex of the simplicial algebra **E** can be given in the form

$$\partial_n(NE_n) = \sum_{I,J} K_I K_J$$

for $\emptyset \neq I, J \subset [n-1] = \{0, 1, \ldots, n-1\}$ with $I \cup J = [n-1]$, where

$$K_I = \bigcap_{i \in I} \operatorname{Ker} d_i \quad \text{and} \quad K_J = \bigcap_{j \in J} \operatorname{Ker} d_j.$$

In general for $n > 4$, we can only prove

$$\sum_{I,J} K_I K_J \subseteq \partial_n(NE_n).$$

Chapter 3 provides a functor from simplicial algebras to crossed complexes, analogous to the group case. We reconsider the particular case of the 'step-by-step' construction so as to define a free crossed resolution of an algebra. We use the above functor and 'Higher Order Peiffer Elements' in order to describe that resolution. Moreover we give several technical results about the particular case of the 'step-by-step' construction.

In chapter 4, we define a notion of 2-crossed modules for commutative algebras clarifying that given in Carrasco's thesis [12] and A.R.Grandjeán and M.J.Vale [24]. The importance of this chapter is to characterise 2-crossed modules by means of the second order Peiffer elements as defined in chapter 2.

The freeness property for this concept is explicitly built in terms of the 'step-by-step' construction. The k-skeleton of that construction induces algebraic models of n-types of the k-skeleton of a free simplicial algebra.

In the final chapter, we recall from [12] the definition of crossed squares of commutative algebras with examples. We use the second order Peiffer elements to determine crossed squares. By taking the idea of Ellis [21] for a construction of a free crossed square, we form a free crossed square for commutative algebras in terms of 2-dimensional data for a free simplicial algebra. We end this chapter by describing a functor from simplicial algebras to crossed n-cubes in order to give various technical results.

Chapter 1

Simplicial Resolutions and Crossed Modules of Algebras

Introduction

Let $\mathbf{k}$ be a fixed commutative ring with $1 \neq 0$ (that is, $\mathbf{k}$ is not trivial). All of the $\mathbf{k}$-algebras discussed herein are assumed to be commutative and associative but we will want to consider ideals and modules to be algebras and so will not be requiring *algebras* to have unit elements. The category of all $\mathbf{k}$-modules will be denoted by **Mod**.

Recall that a *commutative* $\mathbf{k}$-*algebra* (or algebra over $\mathbf{k}$) is a $\mathbf{k}$-module M with an $\mathbf{k}$-bilinear map

$$\begin{array}{ccc} M \times M & \longrightarrow & M \\ (m_1, m_2) & \longmapsto & m_1 m_2 \end{array}$$

satisfying

$$i) \quad m_1 m_2 = m_2 m_1 \quad ii) \quad (m_1 m_2) m_3 = m_1 (m_2 m_3)$$

for all $m_1, m_2, m_3 \in M$. The category of commutative algebras will be denoted by **Alg**.

We commence this chapter by presenting some aspects of the theory of simplicial algebras. In the first section, we recall some general results on simplicial objects. In particular, we restrict attention to simplicial objects in the category of commutative algebras. Section 2 deals with the 'step-by-step' construction of a free simplicial algebras.

The subsequent sections of this chapter contain a summary of much of the elementary theory of crossed modules of commutative algebras. Section 4 is devoted to a definition of crossed modules and some examples. In addition we shall give a few results regarding them. A commutative algebraic version of free crossed modules will be recalled in section 5. The relation between Koszul complexes and free crossed modules is considered in the last section.

1.1 Simplicial Algebras

In this section we recall a few well-known definitions and facts about simplicial algebras and homology modules. For more details regarding this, we refer to the book *Homologie des algèbres commutatives* by M.André [1].

Definition 1.1.1 A simplicial algebra **E** *is a collection of* **k**-*algebras* E_n $(n \in \mathbb{N})$ *together with, for each* $n \geq 0$, **k**-*algebra homomorphisms*

$$\begin{array}{llll} d_i^n : & E_n \longrightarrow E_{n-1} & & 0 \leq i \leq n \neq 0, \\ s_j^n : & E_n \longrightarrow E_{n+1} & & 0 \leq j \leq n, \end{array}$$

which are called face operators and degeneracies respectively. These homomorphisms are required to satisfy the following axioms:

$$\begin{array}{lrcll} 1. & d_i^{n-1}d_j^n & = & d_{j-1}^{n-1}d_i^n & \text{for } 0 \leq i < j \leq n, \\ 2. & s_i^{n+1}s_j^n & = & s_{j+1}^{n+1}s_i^n & \text{for } 0 \leq i \leq j \leq n, \\ 3. & d_i^{n+1}s_j^n & = & s_{j-1}^{n-1}d_i^n & \text{for } 0 \leq i < j \leq n, \\ 4. & d_i^{n+1}s_j^n & = & id & \text{for } i = j \text{ or } i = j+1, \\ 5. & d_i^{n+1}s_j^n & = & s_j^{n-1}d_{i-1}^n & \text{for } 0 \leq j < i-1 \leq n. \end{array}$$

For use in calculation it is often convenient to recall that the above equalities imply the following ones:

$$\begin{array}{lrcll} 1. & d_i d_j & = & d_j d_{i+1} & \text{for } 0 \leq j \leq i \leq n, \\ 2. & s_i s_j & = & s_j s_{i-1} & \text{for } 0 \leq j < i \leq n, \\ 3. & s_i d_j & = & d_j s_{i+1} & \text{for } j \leq i, \\ 4. & s_i d_j & = & d_{j+1} s_i & \text{for } i > j. \end{array}$$

These equations are standard and may be found in [15], [16], [32] and [34].

Elements $x \in E_n$ are called *n-dimensional simplices*. A simplex x is called *degenerate* if $x = s_i(y)$ for some y.

A homomorphism of simplicial algebras **f: E** $\to$ **F** is a set of **k**-algebra homomorphisms $f_n : E_n \to F_n$ commuting with all the face operators, d_i^n, and degeneracy operators, s_j^n, i.e.

$$d_i f_n = f_{n-1} d_i, \quad f_n s_i = s_i f_{n-1}.$$

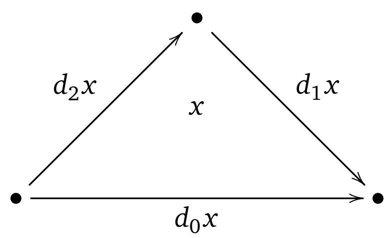
We have thus defined the category of simplicial algebras, which we will denote by **SimpAlg**.

A geometric interpretation of this definition for low dimensions can be thought of as follows:

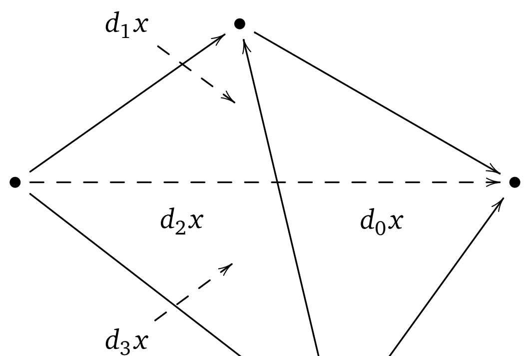
For $n = 0$, a 0-dimensional simplex is simply a point $x \in E_0$ and a 1-dimensional simplex is just, for $x \in E_1$,

$$d_1 x \bullet \xrightarrow{\quad x \quad} \bullet\, d_0 x$$

2-dimensional simplices are just triangles: for $x \in E_2$



and 3-dimensional simplices are just tetrahedra:



and so on.

Definition 1.1.2 A simplicial k-module *is a family of* **k**-*modules* E_n*, for* $n \geq 0$*, and* **k**-*module homomorphisms satisfying the equalities in definition 1.1.1.*

Remark 1.1.3 For any simplicial module **E**, there is an associated chain complex of k-modules. The differentials $\partial_n : E_n \to E_{n-1}$ are defined by

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i^n.$$

By axiom 1 in the definition of simplicial algebras, $\partial_{n+1}\partial_n = 0$. This is thus a chain complex associated to the simplicial module **E**. Hence we can speak of the n^{th} *homology module* $H_n(\mathbf{E})$ of the simplicial **k**-module **E** defined by

$$H_n(\mathbf{E}) = \frac{\mathrm{Ker}\partial_n}{\mathrm{Im}\partial_{n+1}.}$$

Definition 1.1.4 *A simplicial algebra* **E** *is* augmented *by specifying a constant simplicial algebra* $\mathbf{K}(E,0)$ *and a surjective* **k**-*algebra homomorphism,* $f = d_0^0 : E_0 \to E$ *with* $f d_0^1 = f d_1^1 : E_1 \to E$. *An* augmentation *of the simplicial algebra* **E** *is a map*

$$\mathbf{E} \longrightarrow \mathbf{K}(E,0).$$

An augmented simplicial algebra is acyclic *if the corresponding complex is acyclic, i.e.* $H_n(\mathbf{E}) \cong 0$ *for* $n > 0$ *and* $H_0(\mathbf{E}) \cong E$.

Simplicial resolution of an algebra B

Definition 1.1.5 *Let B be a commutative* **k**-*algebra.* A free simplicial resolution of B *consists of a simplicial algebra* **E** *together with an augmentation* $f : E_0 \to B$ *such that* $(\mathbf{E}, f)$ *is acyclic and each* E_n *is free.*

We will summarise André 's construction of a simplicial resolution in section 1.2.

The Moore complex and the homotopy module of a simplicial algebra

Recall that given a simplicial algebra **E**, *the Moore complex* $(\mathbf{NE}, \partial)$ *of* **E** is the chain complex defined by

$$(\mathbf{NE})_n = \bigcap_{i=0}^{n-1} \mathrm{Ker} d_i^n$$

with $\partial_n : NE_n \to NE_{n-1}$ induced from d_n^n by restriction.

The n^{th} *homotopy module* $\pi_n(\mathbf{E})$ of **E** is the nth homology of the Moore complex of **E**, i.e.,

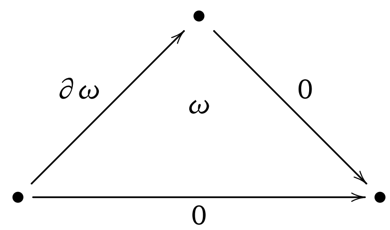
$$\begin{aligned} \pi_n(\mathbf{E}) &\cong H_n(\mathbf{NE}, \partial) \\ &= \bigcap_{i=0}^{n} \mathrm{Ker} d_i^n / d_{n+1}^{n+1}\left(\bigcap_{i=0}^{n} \mathrm{Ker} d_i^{n+1}\right). \end{aligned}$$

The interpretation of **NE** and $\pi_n(\mathbf{E})$ is as follows:

for $n = 1,\ w \in NE_1$,

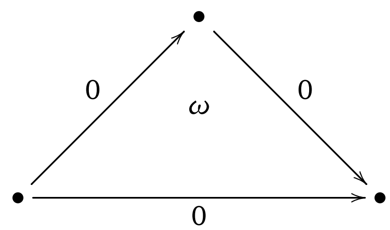
$$\partial\,\omega \bullet \xrightarrow{\ \ \omega\ \ } \bullet\ 0$$

and $w \in NE_2$ looks like



and so on.

Note that: $w \in \mathrm{NE}_2$ is in $\mathrm{Ker}\partial$ if it looks like



whilst it will give the trivial element of $\pi_2(\mathbf{E})$ if there is a 3-simplex x with w on its 3^{rd} face and all other faces zero.

This simple interpretation of the elements of **NE** and $\pi_n(\mathbf{E})$ will 'pay off' later by aiding interpretation of some of the elements in other situations.

By a *k-truncated simplicial algebra*, we mean a simplicial algebra $\mathbf{tr_k E}$ obtained by forgetting dimensions of order $> k$ in a simplicial algebra **E**. We denote the category of k-truncated simplicial algebras by $\mathbf{Tr_k SimpAlg}$. Recall from [16] some facts about the skeleton functor. In the category of algebras, **Alg**, there is a truncation functor

$$\mathbf{tr_k} : \mathbf{SimpAlg} \longrightarrow \mathbf{Tr_k SimpAlg}$$

which admits a right adjoint

$$\mathbf{cosk_k} : \mathbf{Tr_k SimpAlg} \longrightarrow \mathbf{SimpAlg}$$

called the *k-coskeleton functor*, and a left adjoint

$$\mathbf{sk_k} : \mathbf{Tr_k SimpAlg} \longrightarrow \mathbf{SimpAlg},$$

called the *k-skeleton functor*.

Assume given that $\mathbf{tr_k(E)} = \{E_0, E_1, \ldots, E_k\}$ is a k-truncated simplicial algebra. A family of homomorphisms

$$(\delta_0, \ldots, \delta_{k+1}) : X_{k+1} \quad \overset{\delta_{k+1}}{\underset{\delta_0}{\overrightarrow{\vdots}}} \quad E_k \quad \overset{\delta_k}{\underset{\delta_0}{\overrightarrow{\vdots}}} \quad E_{k-1}$$

is said to be the *simplicial kernel* of the family of face homomorphisms $(d_0, \ldots, d_k)$ if it has the following universal property:

given any family $(x_0, \ldots, x_{k+1})$ of $k+2$ homomorphisms

$$Y \quad \overset{x_{k+1}}{\underset{x_0}{\overrightarrow{\vdots}}} \quad E_k$$

which satisfies the equalities $d_i x_j = d_{j-1} x_i$ $(0 \leq i < j \leq k+1)$ with the last part of the truncated simplicial algebra, there exists a unique homomorphism

$$x = \langle x_0, ..., x_{k+1} \rangle : Y \longrightarrow X_{k+1}$$

such that $\delta_i x = x_i$.

Given the simplicial kernel X_{k+1}, the family of homomorphisms

$$(\alpha_{k+1,j}, \ldots, \alpha_{1j}, \alpha_{0j})$$

defined by

$$\alpha_{ij} = \begin{cases} s_{j-1} d_i & \text{if } i < j \\ id & \text{if } i = j \text{ or } \ i = j+1 \\ s_j d_{i-1} & \text{if } i > j+1 \end{cases}$$

satisfies the simplicial identities with the last part of the truncated simplicial algebra; hence there exists a unique $s_j : E_k \to X_{k+1}$ such that $\delta_i s_j = \alpha_{ij}$. The defined $(s_j)_{0 \leq j \leq k}$ form a system of degeneracies and we have now defined a $(k+1)$-truncated simplicial algebra

$$\{E_0, E_1, \ldots, E_k, X_{k+1}\}.$$

By iterating this process we obtain the simplicial algebra

$$\mathbf{cosk_k(tr_k(E))} = \{E_0, E_1, \ldots, E_k, X_{k+1}, X_{k+2}, \ldots\}$$

called the *coskeleton* of the truncated simplicial algebra. If $\mathbf{F}$ is an simplicial algebra, then any truncated simplicial algebra

$$x : \mathbf{tr_k}(\mathbf{E}) \longrightarrow \mathbf{tr_k}(\mathbf{F})$$

extends uniquely to a simplicial map

$$x : \mathbf{E} \longrightarrow \mathbf{cosk_k}(\mathbf{tr_k}(\mathbf{F})).$$

The k-skeleton functor can be constructed by a dual process involving *simplicial cokernels*

$$(s_0, \ldots, s_k) : E_k \quad \overset{s_k}{\underset{s_0}{\xrightarrow[\vdots]{\quad\quad}}} \quad X_{k+1}$$

That is, universal systems of $k+1$ maps verifying $s_i s_j = s_{j+1} s_i$ for $0 \leq i \leq j \leq k-1$. See for details [16] and [2].

The following lemma is due to Conduché [14] for the group case. We give an obvious analogue for the commutative algebra version, but we omit its proof which can be obtained by changing slightly the corresponding result in [22].

Lemma 1.1.6 *Let* $\mathbf{E}$ *be a simplicial algebra. The Moore complex of its k-coskeleton* $\mathbf{cosk_k}(\mathbf{tr_k}(\mathbf{E}))$*is of length* $k+1$*, i.e.,*

$$N(\mathbf{cosk_k}(\mathbf{tr_k}(\mathbf{E})))_i = 0 \qquad \textit{for } i > k+1,$$

and is identical to the Moore complex of $\mathbf{E}$ *in dimension less than* $k+1$*. Moreover*

$$N(\mathbf{cosk_k}(\mathbf{tr_k}(\mathbf{E})))_{k+1} = \mathrm{Ker}(\partial_k : NE_k \longrightarrow NE_{k-1})$$

and the morphism

$$\partial_{k+1} : N(\mathbf{cosk_k}(\mathbf{tr_k}(\mathbf{E})))_{k+1} \longrightarrow N(\mathbf{cosk_k}(\mathbf{tr_k}(\mathbf{E})))_k = NE_k$$

is injective.

1.2 Step By Step Constructions

This section is a brief résumé of how to construct simplicial resolutions. The work depends heavily on a variety of sources, mainly [1], [34], [39]. The reader is referred to the book of André [1] for full details and more references.

1.2.1 Definition and Notation

First recall the following notation and terminology which will be used in the construction of a simplicial resolution.

Let $[n]$ be the ordered set, $[n] = \{0 < 1 < ... < n\}$. We define the following maps: Firstly the injective monotone map $\delta_i^n : [n-1] \to [n]$ is given by

$$\delta_i^n(x) = \begin{cases} x & \text{if} \quad x < i \\ x+1 & \text{if} \quad x \geq i \end{cases}$$

for $0 \leq i \leq n \neq 0$. We display all these maps omitting the superscripts as

$$[0] \overset{\delta_0}{\underset{\delta_1}{\rightrightarrows}} [1] \overset{\delta_0,\ \delta_1}{\underset{\delta_2}{\rightrightarrows\!\!\to}} [2] \overset{\delta_0,\ \delta_1}{\underset{\delta_2,\ \delta_3}{\rightrightarrows\!\!\rightrightarrows}} [3] \ldots$$

On the other hand, an increasing surjective monotone map $\sigma_i^n : [n+1] \to [n]$ is given by

$$\sigma_i^n(x) = \begin{cases} x & \text{if} \quad x \leq i \\ x-1 & \text{if} \quad x > i \end{cases}$$

for $0 \leq i \leq n$. We display them without superscripts as

$$[0] \xleftarrow{\sigma_0} [1] \overset{\sigma_0}{\underset{\sigma_1}{\leftleftarrows}} [2] \overset{\sigma_0,\ \sigma_1}{\underset{\sigma_2}{\leftleftarrows\!\!\leftarrow}} [3] \ldots$$

We denote by $\{m, n\}$ the set of increasing surjective maps $[m] \to [n]$ as used in [34].

1.2.2 Killing Elements in Homotopy Modules

The following section describes the 'step-by-step' construction of André [1].

Let $\mathbf{E}$ be a simplicial algebra and let $k \geq 1$ be fixed. Suppose we are given a set Ω of elements

$$\{x_\lambda : \lambda \in \Lambda\},$$

$x_\lambda \in \pi_{k-1}(\mathbf{E})$, then we can choose a corresponding set of elements $w_\lambda \in NE_{k-1}$ so that

$$x_\lambda = w_\lambda + \partial_k(NE_k).$$

(If $k=1$, then as $NE_0 = E_0$, the condition that $w_\lambda \in NE_0$, is empty.) We want to define a simplicial algebra, $\mathbf{F} = \mathbf{E}[\Omega]$ with a monomorphism

$$\mathbf{i} : \mathbf{E} \longrightarrow \mathbf{F}$$

such that

$$\pi_{k-1}(\mathbf{i}) : \pi_{k-1}(\mathbf{E}) \longrightarrow \pi_{k-1}(\mathbf{F})$$

'kills off' the x_λ's. We do this by adding new indeterminates into NE_k to enlarge it so as to make $\mathbf{i}(w_\lambda) \in \partial NF_k$. More precisely,

1) F_n is a free E_n-algebra,

$$F_n = E_n[y_{\lambda,t}] \text{ with } \lambda \in \Lambda \text{ and } \ t \in \{n,k\}.$$

2) For $0 \leq i \leq n$, the algebra homomorphism $s_i^n : F_n \to F_{n+1}$ is obtained from the homomorphism $s_i^n : E_n \to E_{n+1}$ with the relations

$$s_i^n(y_{\lambda,t}) = y_{\lambda,u} \quad \text{with} \quad u = t\sigma_i^n, \ \ t : [n] \to [k].$$

3) For $0 \leq i \leq n \neq 0$, the algebra homomorphism $d_i^n : F_n \to F_{n-1}$ is obtained from $d_i^n : E_n \to E_{n-1}$ with the relations

$$d_i^n(y_{\lambda,t}) = \begin{cases} y_{\lambda,u} & \text{if the map} \quad u = t\delta_i^n \quad \text{is surjective} \\ t'(w_\lambda) & \text{if} \qquad\qquad u = \delta_k^k t' \\ 0 & \text{if} \qquad\qquad u = \delta_j^k t' \quad \text{with} \ \ j \neq k \end{cases}$$

by extending linearly.

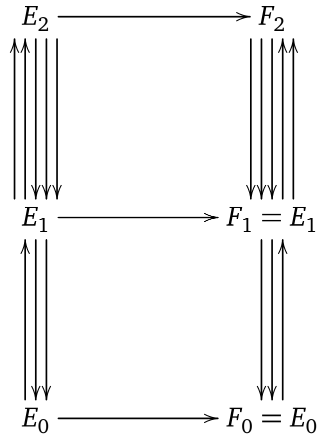
Here $t' : [n-1] \to [k-1]$. It corresponds to a unique algebra homomorphism $t' : E_{k-1} \to E_{n-1}$, c.f. M.André [1].

We now examine this construction for a single element to see what it does:

Example 1.2.1 *To explain the construction, we will see how to kill a single element $x \in \pi_1(\mathbf{E})$ (so $k=2$). Pick a $w \in NE_1$ so that*

$$x = \bar{w} = w + \partial_2(NE_2) \in \pi_1(\mathbf{E}).$$

We thus have the following diagram



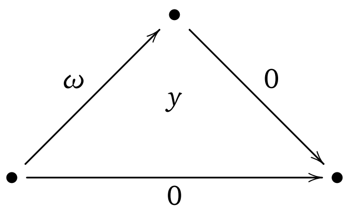
and we need a $y \in NE_2$ *with*

$$w = \partial(y) = d_2(y) \text{ with } \bar{w} = w + \partial_2(NE_2) \in \pi_1(\mathbf{E})$$

and hence we add a new indeterminate y *(which will be non-degenerate) into* E_2 *to form*

$$F_2 = E_2[y] \text{ with } \quad d_0(y) = d_1(y) = 0 \textit{ and } d_2(y) = w.$$

Geometrically for $k = 2$*,*



which implies

$$\mathbf{i}(\bar{w}) = \mathbf{i}(w + \partial NE_2) = 0$$

as required. We cannot stop here as the images of y *under* s_0, s_1, s_2 *are not yet defined.*

For the next step we build F_3 *so as to receive the degenerate images of* y*, i.e.,*

$$F_3 = E_3[y_t],$$

where $t : [3] \to [2]$*. So there are three degenerate images corresponding to* $s_0(y), s_1(y), s_2(y)$*. We set*

$$s_0(y) = y_{\sigma(0)},\ s_1(y) = y_{\sigma(1)},\ s_2(y) = y_{\sigma(2)},$$

and also need to construct the face operators

$$d_0, d_1, d_2, d_3 : F_3 \longrightarrow F_2$$

but these are determined in advance since

$$d_0 s_i(y) = s_{i-1} d_0(y) = 0 \ \ unless \ \ i = 0$$

in which case $d_0 s_0(y) = y$. *We then define recursively the higher dimensional images of* y. *In the formula given above this is done all together (following André* [1]*).*

Remark 1.2.2 In the 'step-by-step' construction of simplicial resolutions, there are the subsequent properties:

i) $F_n = E_n$ for $n < k$,

ii) $F_k =$ a free E_k-algebra over a set of non-degenerate indeterminates, all of whose faces are zero except the k^{th},

iii) F_n is a free E_n-algebra over the degenerate elements for $n > k$.

Later on, this is called the k-skeleton of a resolution.

We have immediately the following result, as expected.

Proposition 1.2.3 *The inclusion of simplicial algebras* $\mathbf{E} \hookrightarrow \mathbf{F}$*, where* $\mathbf{F} = \mathbf{E}[\Omega]$*, induces the homomorphism*

$$\pi_n(\mathbf{E}) \longrightarrow \pi_n(\mathbf{F}).$$

For $n < k - 1$,

$$\pi_n(\mathbf{E}) \cong \pi_n(\mathbf{F})$$

and for $n = k - 1$*, this homomorphism is an epimorphism with kernel generated by elements of the form* $\bar{w}_\lambda = w_\lambda + \partial_k N E_k$.

1.2.3 Constructing Simplicial Resolutions

The following result is due to André [1].

Theorem 1.2.4 *If* B *is a commutative* $\mathbf{k}$*-algebra, then it has a simplicial resolution* $\mathbf{R}$.

Proof: The repetition of the above construction will give us the simplicial resolution of an algebra.

Let B be a commutative **k**-algebra and let E be a free **k**-algebra. We denote by $\mathbf{K}(E,0)$ the simplicial algebra which in each dimension is equal to E and in which each face and degeneracy map is the identity. We describe the zero step of the construction. It consists of the choice of a free **k**-algebra E and a surjection $f:E \to B$ which gives an isomorphism $E/\mathrm{Ker}f \cong B$ as **k**-algebras. Then we form the trivial simplicial algebra $\mathbf{E}^{(0)}$ for which in every degree n, $E_n = E$ and $d_i^n = \mathrm{id} = s_j^n$ for all i,j. Thus $\mathbf{E}^{(0)} = \mathbf{K}(E,0)$ and $\pi_0(\mathbf{E}^{(0)}) = E$. Now choose a set Ω^0 of generators of the ideal $I = \mathrm{Ker}(E \stackrel{f}{\to} B)$, and obtain the simplicial algebra in which $E_1^{(1)} = E[\Omega^0]$ and for $n > 1$, $E_n^{(1)}$ is a free E_n-algebra over the degenerate elements. This simplicial algebra is denoted by $\mathbf{E}^{(1)}$ and will be called the *1-skeleton of a simplicial resolution of an algebra B*.

The consequent steps depend on the choice of sets, Ω^0, $\Omega^1, \Omega^2, \ldots, \Omega^k, \ldots$ Let $\mathbf{E}^{(k)}$ be the simplicial algebra constructed after k steps, the k-skeleton of the resolution. The set Ω^k is formed by elements w of $E_k^{(k)}$ with $d_i^k(w) = 0$ for $0 \leq i \leq k$ and whose images $\bar{w}$ in $\pi_k(\mathbf{E}^{(k)})$ generate that module over $E_k^{(k)}$.

Finally we have inclusions of simplicial algebras

$$\mathbf{E} = \mathbf{E}^{(0)} \subseteq \mathbf{E}^{(1)} \ldots \subseteq \mathbf{E}^{(k-1)} \subseteq \mathbf{E}^{(k)} \subseteq \ldots$$

and in passing to the inductive limit (colimit), we obtain an acyclic free simplicial **k**-algebra **R** with $R_n = E_n^{(k)}$ if $n \leq k$. **R** is thus a simplicial resolution of **k**-algebra B. The proof of theorem is completed. □

Remark 1.2.5 A variant of the step-by-step construction gives:

if **A** is a simplicial algebra, then there exists a free simplicial algebra **E** and an epimorphism

$$\mathbf{E} \longrightarrow \mathbf{A}$$

which induces isomorphisms on all homotopy modules. The details are omitted.

We have not talked here about the homotopy of simplicial algebra morphisms, and so will not discuss homotopy invariance of this construction for which see André [1].

1.3 Crossed Modules

J.H.C.Whitehead (1949) [43] described crossed modules in various contexts especially in his investigations into the algebraic structure of relative homotopy groups. In this section, we introduce the definition and elementary theory of crossed modules of commutative algebras given by T.Porter, [36]. More details about this may be found in [42], [18] and [19], see also [4].

We recall that if M and R are commutative algebras, a map

$$\begin{array}{rcl} R\times M & \longrightarrow & M \\ (r,m) & \longmapsto & r\cdot m, \end{array}$$

is a commutative action if and only if

1. $k(r\cdot m)=(kr)\cdot m=r\cdot(km)$,
2. $r\cdot(m+m')=r\cdot m+r\cdot m'$,
3. $(r+r')\cdot m=r\cdot m+r'\cdot m$,
4. $r\cdot(mm')=(r\cdot m)m'=m(r\cdot m')$,
5. $(rr')\cdot m=r(r'\cdot m)$,

for all $k\in\mathbf{k}$, $m,m'\in M$, $r,r'\in R$.

Throughout this thesis we denote an action of $r\in R$ on $m\in M$ by $r\cdot m$.

Definition 1.3.1 *Let R be a* **k***-algebra with identity.* A pre-crossed module of commutative algebras *is an R-algebra C, together with a commutative action of R on C and an R-algebra morphism*

$$\partial : C\longrightarrow R,$$

such that for all $c\in C, r\in R$

$$CM1)\qquad \partial(r\cdot c) \;=\; r\partial c.$$

This is a crossed R-module *if in addition, for all* $c,c'\in C$,

$$CM2)\qquad \partial c\cdot c' \;=\; cc'.$$

The last condition is called *the Peiffer identity*. We denote such a crossed module by (C,R,∂). Clearly any crossed module is a pre-crossed module.

Definition 1.3.2 A morphism of crossed modules *from* (C,R,∂) *to* (C',R',∂') *is a pair of* **k**-*algebra morphisms,*

$$\theta : C \longrightarrow C', \quad \psi : R \longrightarrow R',$$

such that

$$\theta(r \cdot c) = \psi(r) \cdot \theta(c) \text{ and } \partial'\theta(c) = \psi\partial(c).$$

In this case, we shall say that θ is a crossed R-module morphism if $R = R'$ and ψ is the identity. We therefore can define the category of crossed modules denoting it as **XMod**.

1.3.1 Examples

Example 1.3.3 *Let I be any ideal of a* **k**-*algebra R. Consider an inclusion map*

$$inc. : I \longrightarrow R.$$

Then $(I,R,inc.)$ is a crossed module. Conversely given any crossed R-module $\partial\colon C \to R$, one can easily verify that $\partial C = I$ is an ideal in R.

Example 1.3.4 *Let M be any R-module. It can be considered as an R-algebra with zero multiplication, and then $\mathbf{0} : M \to R$ is a crossed R-module by $\mathbf{0}(c) \cdot c' = 0c' = 0 = cc'$, for all $c, c' \in C$.*

Conversely, given any crossed module $\partial : C \to R$, then $\mathrm{Ker}\partial$ is an $R/\partial C$-module. For this, see Proposition 1.3.6.

Lemma 1.3.5 *Assume given a simplicial algebra* **E** *and a simplicial ideal* **I***. The inclusion*

$$inc. : \mathbf{I} \hookrightarrow \mathbf{E},$$

induces a map

$$\partial : \pi_0(\mathbf{I}) \longrightarrow \pi_0(\mathbf{E}),$$

and **E** *acting on* **I** *by multiplication induces an action of $\pi_0(\mathbf{E})$ on $\pi_0(\mathbf{I})$. Then $(\pi_0(\mathbf{I}), \pi_0(\mathbf{E}), \partial)$ is a crossed module.*

Proof: CM1) For all $e \in \mathbf{E}$,

$$\begin{aligned} \partial([e]\cdot[i]) &= [ei], \\ &= [e][inc.(i)], \\ &= [e]\partial([i]). \end{aligned}$$

CM2) For all $i, i' \in \mathbf{I}$,

$$\begin{aligned} \partial([i])\cdot[i'] &= [inc.(i)\cdot i'], \\ &= [ii'], \\ &= [i][i']. \end{aligned}$$

□

Any crossed module can be obtined as π_0 of an ideal inclusion, $\mathbf{I} \hookrightarrow \mathbf{E}$, of simplicial algebras but we will not include a proof here. This generalises easily to the crossed n-cubes of chapter 5.

The following result is due to N.M.Shammu [42].

Proposition 1.3.6 *If (C,R, ∂) is a crossed R-module, then*

i) Ker∂ *is a central ideal of C,*

ii) both C/C^2 *and* Ker∂ *have natural* $R/\partial C$*-module structure.*

Proof: i) Since, for $c \in C,\ a \in \mathrm{Ker}\partial$,

$$ac = \partial a \cdot c = 0c = 0 = c0 = c \cdot \partial a = ca$$

as required.

ii) It is enough to show that ∂C acts trivially on Ker∂ and C/C^2.

For $a \in \mathrm{Ker}\partial,\ \partial c \in \partial C,$ by $\partial c \cdot a = ca = c \cdot \partial a = c0 = 0,\ \partial C$ acts trivially on $\mathrm{Ker}\partial$.

For $\partial c \in \partial C,\ c' + C \in C/C^2$, we obtain the following

$$\begin{aligned} \partial c \cdot (c' + C^2) &= \partial c \cdot c' + C^2 \\ &= cc' + C^2 \\ &= 0, \end{aligned}$$

so ∂C acts trivially on C/C^2. Hence we can unambiguously define maps

$$\begin{array}{rclrcl} R/\partial C \times \mathrm{Ker}\partial & \longrightarrow & \mathrm{Ker}\partial & R/\partial C \times C/C^2 & \longrightarrow & C/C^2 \\ (r + \partial c, a) & \longmapsto & ra & (r + \partial c, c + C^2) & \longmapsto & rc + C^2 \end{array}$$

and it is routine to check that this turns the abelian groups $\mathrm{Ker}\partial$ and C/C^2 into R/∂C-modules. Thus $\mathrm{Ker}\partial$ and C/C^2 have $R/\partial C$-module structure. □

1.4 Free Crossed Modules

The notion of a free crossed module of commutative algebras was earlier described by E.R. Aznar [4]. In this section, we recall how to form a free crossed module.

Definition 1.4.1 *Let (C,R,∂) be a crossed module, let Y be a set and let $\nu : Y \to C$ be a function, then (C,R,∂) is said to be a* free crossed R-module with basis ν *or, alternatively,* on the function $\partial\nu : Y \to R$ *if for any crossed R-module (C',R,∂') and function $\nu' : Y \to C'$ such that $\partial'\nu' = \partial\nu$, there is a unique morphism*

$$\phi : (C,R,\partial) \to (C',R,\partial')$$

such that $\phi\nu = \nu'$.

The crossed module (C,R,∂) is *totally free* if R is a free algebra. On replacing 'crossed' by 'pre-crossed' in the above definition of a (totally) free crossed module, we obtain the appropriate definition of a *(totally) free pre-crossed module*. The following proof of the result is taken from T.Porter [36].

Theorem 1.4.2 *A free crossed module R-module (C,R,∂) exists on any function $f : Y \to R$ with codomain R.*

Proof: Given a function from a set Y to the **k**-algebra R, $f : Y \to R$, consider $E = R^+[Y]$, the positively graded part of the polynomial ring on Y so that R acts on E by multiplication. The function f induces a morphism of R-algebras

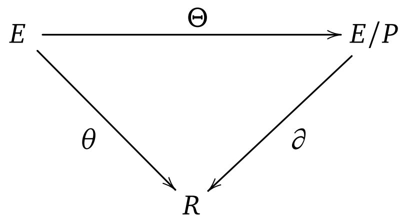
$$\theta : R^+[Y] \longrightarrow R$$

given by $\theta(y) = f(y)$.

Let (A,R,δ) be any crossed module. We suppose given $\omega : Y \to A$ such that $\delta\omega = f$. Let P be the ideal of $R^+[Y]$ generated by all elements of the form

$$P = \{pq - \theta(p)q : p,q \in R^+[Y]\}.$$

One readily sees that $\theta(P) = 0$. If we put $C = E/P$, we then have the natural commutative diagram

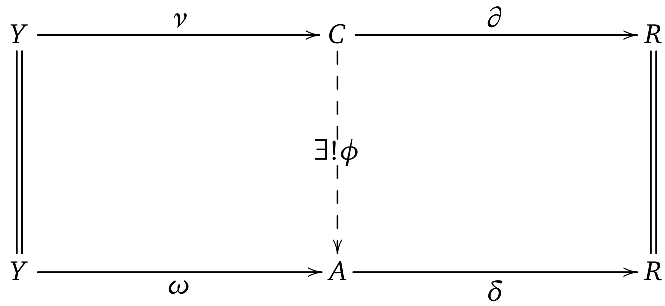


Here Θ is the canonical quotient map of algebras. We now show that the Peiffer condition can be satisfied as follows;

For all $y_1 + P, y_2 + P \in C$,

$$\begin{aligned}
\partial(y_1 + P) \cdot (y_2 + P) &= \theta y_1(y_2 + P) \\
&= \theta y_1 y_2 + P \\
&\equiv y_1 y_2 + P \mod P \\
&= (y_1 + P)(y_2 + P).
\end{aligned}$$

There exists a unique morphism $\phi : C \to A$ given by $\phi(y + P) = w(y)$ such that $\delta\phi = \partial$, i.e.



Hence $(C(f), R, \partial)$ is the required free crossed module R-module on f. □

Remark 1.4.3 Later on, P will be denoted by P_1 and will be called the first order Peiffer ideal, as our aim is to identify higher order versions of the Peiffer elements.

1.5 Relations Between Free Crossed Modules and Koszul Complexes

1.5.1 DEFINITION

Let M be an $\mathbf{k}$-module and let $\varphi : M \to \mathbf{k}$ be a homomorphism of $\mathbf{k}$-modules. We define $\mathbf{K}(\varphi)$ by setting $K_p(\varphi) = \Lambda^p(M)$, the pth exterior power of M, for $p \geq 0, K_p(\varphi) = 0$ for $p < 0$. We have the differential :

$$d_p : \Lambda^p(M) \longrightarrow \Lambda^{p-1}(M)$$

given by the formula

$$d_p(u_1 \wedge \ldots \wedge u_p) = \sum_{j=1}^{p} (-1)^{j-1} \varphi(u_j) u_1 \wedge \ldots \wedge u_{j-1} \wedge u_{j+1} \wedge \cdots \wedge u_p.$$

A simple calculation shows that $d_{p-1} d_p = 0$ for all p, and therefore $\mathbf{K}(\varphi)$ is a complex of $\mathbf{k}$-modules.

Definition 1.5.1 *The complex* $\mathbf{K}(\varphi)$ *described above is called the Koszul complex of the homomorphism* $\varphi : M \to \mathbf{k}$. *If* $\mathbf{x}$ *denotes a sequence of elements* $x_1, \ldots, x_n$ *of the ring* $\mathbf{k}$ *and if F is a free module of rank n with basis* $e_1, \ldots, e_n$, *then the Koszul complex* $\mathbf{K}(\varphi)$ *of the homomorphism* $\varphi : F \to \mathbf{k}$ *for which* $\varphi(e_i) = x_i, 1 \leq i \leq n$ *, is also denoted by* $\mathbf{K}(\mathbf{x})$.

If $e_1, \ldots, e_n$ constitute a basis of the module F, then a basis of the module $\Lambda^p(F)$ consists of elements of the form $e_{i_1} \wedge \ldots \wedge e_{i_p}$ for all the sequences of positive integers subject to $1 \leq i_1 < \ldots < i_p \leq n$. Thus the Koszul complex $\mathbf{K}(\mathbf{x})$ is a finite complex of free modules.

We state relations between free crossed module and the Koszul complex from Porter [36] that were already hinted at in Lichtenbaum and Schlessinger [31].

Proposition 1.5.2 *If (C,R, ∂) is a free crossed module R-module on a function* $f : Y \to R$, *with* $Y = \{y_1, \ldots y_n\}$, *then there is a natural isomorphism*

$$C \cong R^n / \mathrm{Im} d,$$

where $d : \Lambda^2 R^n \to R^n$ *is the Koszul differential.*

Proof: See T.Porter [36]. □

Chapter 2

Higher Order Peiffer Elements

Introduction

In this chapter we show the following results:

Let $\mathbf{E}$ be a simplicial commutative algebra with Moore complex $\mathbf{NE}$ and for $n > 1$, let D_n be the ideal generated by the degenerate elements in dimension n. If $E_n = D_n$, then

$$\partial_n(NE_n) = \partial_n(I_n) \quad \text{for all } n > 1,$$

where I_n is an ideal in E_n generated by a fairly small explicitly given set of elements.

If $n = 2, 3$ or 4, then the image of the Moore complex of the simplicial algebra $\mathbf{E}$ can be given in the form

$$\partial_n(NE_n) = \sum_{I,J} K_I K_J$$

where $\emptyset \neq I, J \subset [n-1] = \{0, 1, \ldots, n-1\}$ with $I \cup J = [n-1]$, and where

$$K_I = \bigcap_{i \in I} \mathrm{Ker} d_i \;\; \text{and} \;\; K_J = \bigcap_{j \in J} \mathrm{Ker} d_j.$$

In general for $n > 4$, we can only prove

$$\sum_{I,J} K_I K_J \subseteq \partial_n(NE_n)$$

and suspect the opposite inclusion holds as well.

2.1 Definition and Notation

We firstly recall the following notation and terminology from P.Carrasco and A.M.Cegarra [13].

For the ordered set $[n] = \{0 < 1 < \ldots < n\}$, let $\alpha_i^n : [n+1] \to [n]$ be the increasing surjective map given by

$$\alpha_i^n(j) = \begin{cases} j & \text{if} \quad j \le i \\ j-1 & \text{if} \quad j > i. \end{cases}$$

Let $S(n, n-r)$ be the set of all monotone increasing surjective maps from $[n]$ to $[n-r]$. This can be generated from the various α_i^n by composition. The composition of these generating maps is subject to the following rule

$$\alpha_j \alpha_i = \alpha_{i-1}\alpha_j, \qquad j < i.$$

This implies that every element $\alpha \in S(n, n-r)$ has a unique expression as

$$\alpha = \alpha_{i_1} \circ \alpha_{i_2} \circ \ldots \circ \alpha_{i_r}$$

with $0 \le i_1 < i_2 < \ldots < i_r \le n-1$, where the indices i_k are the elements of $[n]$ such that

$$\{i_1, \ldots, i_r\} = \{i : \alpha(i) = \alpha(i+1)\}.$$

We thus can identify $S(n, n-r)$ with the set

$$\{(i_r, \ldots, i_1) \ : \ 0 \le i_1 < i_2 < \ldots < i_r \le n-1\}.$$

In particular, the single element of $S(n, n)$, defined by the identity map on $[n]$, corresponds to the empty 0-tuple ( ) denoted by $\emptyset_n$. Similarly the only element of $S(n, 0)$ is $(n-1, n-2, \ldots, 0)$. For all $n \ge 0$, let

$$S(n) = \bigcup_{0 \le r \le n} S(n, n-r).$$

We say that $\alpha = (i_r, \ldots, i_1) < \beta = (j_s, \ldots, j_1)$ in $S(n)$

$$\begin{aligned} &\text{if} \quad i_1 = j_1, \ldots, i_k = j_k \quad \text{but} \quad i_{k+1} > j_{k+1} \ \ (k \ge 0) \text{ or} \\ &\text{if} \quad i_1 = j_1, \ldots, i_r = j_r \quad \text{and} \quad r < s. \end{aligned}$$

This makes $S(n)$ an ordered set. For instance, the order in $S(2)$ and in $S(3)$ are respectively:

$$\begin{aligned} S(2) &= \{\emptyset_2 < (1) < (0) < (1,0)\}; \\ S(3) &= \{\emptyset_3 < (2) < (1) < (2,1) < (0) < (2,0) < (1,0) < (2,1,0)\}. \end{aligned}$$

We also define $\alpha \cap \beta$ as the set of indices which belong to both of them.

2.2 The Semidirect Decomposition of a Simplicial Algebra

The fundamental idea behind this can be found in Conduché [14].A detailed investigation of it for the case of a simplicial group is given in Carrasco and Cegarra [13]. The algebra case of that structure is also done in Carrasco's thesis [12].

Definition 2.2.1 *Let M be a $\mathbf{k}$-algebra with $M_1, M_2, \ldots, M_n, n \geq 2$, subalgebras of M.*

The $\mathbf{k}$-algebra M is said to be an n-semidirect product of $M_1, M_2, \ldots, M_n$ if

$$\begin{array}{llll} (i) & M_1 + \ldots + M_s & \text{is an ideal of } M \text{ for } 1 \leq s \leq n, \\ (ii) & M_1 + \ldots + M_n & = M, \\ (iii) & (M_1 + \ldots + M_s) \cap M_t & = 0 \text{ for } 1 \leq s < t \leq n. \end{array}$$

We shall denote this $M = M_1 \rtimes M_2 \rtimes \ldots \rtimes M_n$. Any element can be uniquely expressed as $m_1 + \ldots + m_n$ with $m_i \in M_i$. For this, see P.Carrasco and A.M.Cegarra (1991) [13].

In the following, we see how the n-semidirect product algebras occur in a simplicial algebra.

Lemma 2.2.2 *Let $\mathbf{E}$ be a simplicial algebra. Then E_n can be decomposed as a semidirect product:*

$$E_n \cong \mathrm{Ker} d_n^n \rtimes s_{n-1}^{n-1}(E_{n-1}).$$

Proof: The isomorphism can be defined as follows:

$$\begin{array}{rccl} \theta : & E_n & \longrightarrow & \mathrm{Ker} d_n^n \rtimes s_{n-1}^{n-1}(E_{n-1}) \\ & e & \longmapsto & (e - s_{n-1} d_n e, s_{n-1} d_n e). \end{array}$$

□

Since we have the isomorphism between E_n and $\mathrm{Ker} d_n \rtimes s_{n-1} E_{n-1}$, we can repeat this process as often as necessary to get each of the E_n as a multiple semidirect product of degeneracies of terms in the Moore complex. In fact, let $\mathbf{K}$ be the simplicial algebra defined by

$$K_n = \mathrm{Ker} d_{n+1}^{n+1}, \quad d_i^n = d_i^{n+1} |_{\mathrm{Ker} d_{n+1}^{n+1}} \quad \text{and} \quad s_i^n = s_i^{n+1} |_{\mathrm{Ker} d_{n+1}^{n+1}} .$$

Since $d_{n-1}^{n-1} d_i^n = d_i^{n-1} d_n^n$, for all $i \leq n-1, \mathrm{Ker} d_n^n$ is mapped to $\mathrm{Ker} d_{n-1}^{n-1}$ by all the morphisms d_i^n, $i \leq n-1$. Further, as $d_{n+1}^{n+1} s_i^n = s_i^{n-1} d_n^n$, for $i \leq n-2$, we have that s_i^{n+1} maps $\mathrm{Ker} d_n^n$ to

$\mathrm{Ker} d^{n+1}_{n+1}$. Applying lemma 2.2.2 above, to E_{n-1} and to K_{n-1}, gives

$$\begin{aligned} E_n &\cong \mathrm{Ker} d_n \rtimes s_{n-1}E_{n-1} \\ &= \mathrm{Ker} d_n \rtimes s_{n-1}(\mathrm{Ker} d_{n-1} \rtimes s_{n-2}E_{n-2}) \\ &= K_{n-1} \rtimes (s_{n-1}\mathrm{Ker} d_{n-1} \rtimes s_{n-1}s_{n-2}E_{n-2}). \end{aligned}$$

Since **K** is a simplicial algebra, we have got the following

$$\begin{aligned} \mathrm{Ker} d_n = K_{n-1} &\cong \mathrm{Ker} d^K_{n-1} \rtimes s_{n-2}K_{n-2} \\ &= (\mathrm{Ker} d_{n-1} \cap \mathrm{Ker} d_n) \rtimes s_{n-2}\mathrm{Ker} d_{n-1} \end{aligned}$$

and this enables us to write

$$E_n = ((\mathrm{Ker} d^n_{n-1} \cap \mathrm{Ker} d^n_n) \rtimes s_{n-2}(\mathrm{Ker} d^{n-1}_{n-1})) \rtimes (s_{n-1}(\mathrm{Ker} d^{n-1}_{n-1}) \rtimes s_{n-1}s_{n-2}(E_{n-2})).$$

We can thus decompose E_n as follows:

Proposition 2.2.3 *If* **E** *is a simplicial algebra, then for any* $n \geq 0$

$$\begin{aligned} E_n \cong\ & (\ldots(NE_n \rtimes s_{n-1}NE_{n-1}) \rtimes \ldots \rtimes s_{n-2}\ldots s_0NE_1) \rtimes \\ & (\ldots(s_{n-2}NE_{n-1} \rtimes s_{n-1}s_{n-2}NE_{n-2}) \rtimes \ldots \rtimes s_{n-1}s_{n-2}\ldots s_0NE_0). \end{aligned}$$

The bracketting and the order of terms in this multiple semidirect product are generated by the sequence:

$$\begin{aligned} E_1 &\cong NE_1 \rtimes s_0NE_0 \\ E_2 &\cong (NE_2 \rtimes s_1NE_1) \rtimes (s_0NE_1 \rtimes s_1s_0NE_0) \\ E_3 &\cong ((NE_3 \rtimes s_2NE_2) \rtimes (s_1NE_2 \rtimes s_2s_1NE_1)) \rtimes \\ &\qquad ((s_0NE_2 \rtimes s_2s_0NE_1) \rtimes (s_1s_0NE_1 \rtimes s_2s_1s_0NE_0)). \end{aligned}$$

and

$$\begin{aligned} E_4 \cong\ & (((NE_4 \rtimes s_3NE_3) \rtimes (s_2NE_3 \rtimes s_3s_2NE_2)) \rtimes \\ & ((s_1NE_3 \rtimes s_3s_1NE_2) \rtimes (s_2s_1NE_2 \rtimes s_3s_2s_1NE_1))) \rtimes \\ & s_0(\text{decomposition of } E_3). \end{aligned}$$

Note that the term corresponding to $\alpha = (i_r, \ldots, i_1) \in S(n)$ is $s_\alpha(NE_{n-\#\alpha}) = s_{i_r\ldots i_1}(NE_{n-\#\alpha}) = s_{i_r}\ldots s_{i_1}(NE_{n-\#\alpha})$, where $\#\alpha = r$. Hence any element $x \in E_n$ can be written in the form

$$x = y + \sum_{\alpha \in S(n)} s_\alpha(x_\alpha) \qquad \text{with } y \in NE_n \text{ and } x_\alpha \in NE_{n-\#\alpha}.$$

2.3 Higher Order Peiffer Elements

The following lemma is noted by P.Carrasco [12].

Lemma 2.3.1 *For a simplicial algebra* **E***, there is a bijection between*

$$NE_n = \bigcap_{i=0}^{n-1} \mathrm{Ker} d_i \quad \text{and} \quad \overline{NE}_n^{(r)} = \bigcap_{i \neq r} \mathrm{Ker} d_i$$

in E_n.

Proof: The bijection is given as follows;

$$\begin{array}{rccl} \varphi : & NE_n & \longrightarrow & \overline{NE}_n^{(r)} \\ & e & \longmapsto & \varphi(e) = e - \sum\limits_{k=0}^{n-r-1} (-1)^{k+1} s_{r+k} d_n e. \end{array}$$

It is easy to check that this is a bijection. □

Note that φ is not a homomorphism, but it is additive.

In particular we have:

Lemma 2.3.2 *If E is a simplicial algebra, then there is a bijection*

$$\varphi' : \mathrm{Ker} d_0 \cap \ldots \cap \mathrm{Ker} d_{n-1} \longrightarrow \mathrm{Ker} d_1 \cap \ldots \cap \mathrm{Ker} d_n.$$

Proof: The bijection φ' can be defined by

$$\varphi'(x) = x + \sum_{i=0}^{n-1} (-1)^{n-i} s_{n-i-1} d_n(x).$$

From a direct calculation, φ' is injective and surjective. □

Lemma 2.3.3 *Given a simplicial algebra* **E***, then we have the following*

$$d_n(NE_n) = d_r(\overline{NE}_n^{(r)}).$$

Proof: It is easy to see that, for all elements of the form

$$e - \sum_{k=0}^{n-r-1} (-1)^{k+1} s_{r+k} d_n e$$

of $\overline{NE}_n^{(r)}$ with $e \in NE_n$, one gets

$$d_r\left(e - \sum_{k=0}^{n-r-1}(-1)^{k+1}s_{r+k}d_n e\right) = d_n e$$

as required, but by the proof of lemma 2.3.1 all elements of $\overline{NE}_n^{(r)}$ have this form. □

Proposition 2.3.4 *Let* **E** *be a simplicial algebra, then for $n \geq 2$ and $I, J \subseteq [n-1]$ with $I \cup J = [n-1]$*

$$(\bigcap_{i\in I}\text{Ker}d_i)(\bigcap_{j\in J}\text{Ker}d_j) \subseteq \partial_n NE_n.$$

Proof: For any $J \subset [n-1], J \neq \emptyset$, let r be the smallest element of J. If $r = 0$, then replace J by I and restart and if $0 \in I \cap J$, then redefine r to be the smallest nonzero element of J. Otherwise continue. Let $e_0 \in \bigcap_{j\in J}\text{Ker}d_j$ and $e_1 \in \bigcap_{i\in I}\text{Ker}d_i$, one obtains

$$d_i(s_{r-1}e_0 s_r e_1) = 0 \text{ for } i \neq r$$

and hence $s_{r-1}e_0 s_r e_1 \in \overline{NE}_n^{(r)}$. It follows that

$$e_0 e_1 = d_r(s_{r-1}e_0 s_r e_1) \in d_r(\overline{NE}_n^{(r)}) = d_n NE_n \quad \text{by the previous lemma,}$$

and this implies

$$(\bigcap_{i\in I}\text{Ker}d_i)(\bigcap_{j\in J}\text{Ker}d_j) \subseteq \partial_n NE_n.$$

□

We will denote

$$K_I = \bigcap_{i\in I}\text{Ker}d_i \quad \text{and} \quad K_J = \bigcap_{j\in J}\text{Ker}d_j.$$

Proposition 2.3.4 trivially implies: let **E** be a simplicial algebra with Moore complex **NE**, then

$$\sum_{I,J} K_I K_J \subseteq \partial_n NE_n$$

for $\emptyset \neq I, J \subset [n-1]$ and $I \cup J = [n-1]$.

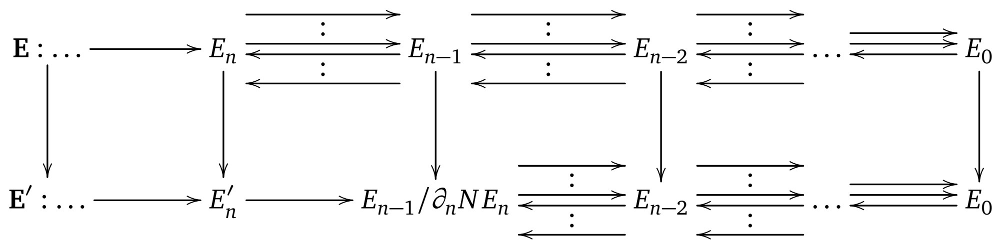
Example 2.3.5 *Let us illustrate this inclusion for $n = 2$. We suppose that $x, y \in NE_1 = \mathrm{Ker}d_0$ so that $(s_0d_1y - y) \in \mathrm{Ker}d_1$. Note that*

$$x(s_0d_1y - y) = d_2(s_1x(s_0y - s_1y))$$

which corresponds to a first order Peiffer element. These elements vanish for all x, y if and only if $\partial_1 : NE_1 \to NE_0$ is a crossed module. Also $\mathrm{Ker}d_0\mathrm{Ker}d_1 \subseteq \partial_2(NE_2)$.

Note that: $\partial_n(NE_n)$ is an ideal in E_{n-1}. In fact, let $x \in NE_n$ and $z \in E_{n-1}$. Define $w = s_{n-1}(z)x$. Then $d_i(w) = 0$ for $i \le n-1$, hence $w \in NE_n$ and $d_n(w) = zd_n(x)$ and so $z\partial_n(x) \in \partial_n(NE_n)$ as required.

Corollary 2.3.6 *Let $\mathbf{E}$ be a simplicial algebra and let $\mathbf{E}'$ be the corresponding truncated simplicial algebra of order n, so we have the canonical morphism:*



Then $\mathbf{E}'$ satisfies the following property:

For all nonempty sets of indices ($I \neq J$) $I, J \subset [n-1]$ with $I \cup J = [n-1]$,

$$(\bigcap_{j\in J} \mathrm{Ker}d_j^{n-1})(\bigcap_{i\in I} \mathrm{Ker}d_i^{n-1}) = 0.$$

Proof: Since $\partial_n NE'_n = 0$, this follows from proposition 2.3.4. □

In the following we will define an ideal I_n. First of all we recall from P.Carrasco [12] the construction of a useful family of **k**-linear morphisms. We define a set $P(n)$ consisting of pairs of elements (α, β) from $S(n)$ with $\alpha \cap \beta = \emptyset$, where $\alpha = (i_r, \ldots, i_1), \beta = (j_s, ..., j_1) \in S(n)$. The **k**-linear morphisms that we will need,

$$\{C_{\alpha,\beta} : NE_{n-\#\alpha} \otimes NE_{n-\#\beta} \longrightarrow NE_n : (\alpha, \beta) \in P(n), \quad n \ge 0\}$$

are given as composites by the diagrams

$$\begin{array}{ccc} NE_{n-\#\alpha}\otimes NE_{n-\#\beta} & \xrightarrow{C_{\alpha,\beta}} & NE_n \\ \Big\downarrow{\scriptstyle s_\alpha\otimes s_\beta} & & \Big\uparrow{\scriptstyle p} \\ E_n\otimes E_n & \xrightarrow{\mu} & E_n \end{array}$$

where

$$s_\alpha = s_{i_r}\dots s_{i_1} : NE_{n-\#\alpha} \longrightarrow E_n\ ,\quad s_\beta = s_{j_s}\dots s_{j_1} : NE_{n-\#\beta} \longrightarrow E_n,$$

$p : E_n \to NE_n$ is defined by composite projections $p = p_{n-1}\dots p_0$, where

$$p_j = 1 - s_j d_j \qquad \text{with} \quad j = 0, 1, \dots n-1$$

and we denote the multiplication by $\mu : E_n \otimes E_n \to E_n$. Thus

$$\begin{aligned} C_{\alpha,\beta}(x_\alpha \otimes y_\beta) &= p\mu(s_\alpha \otimes s_\beta)(x_\alpha \otimes y_\beta) \\ &= p(s_\alpha(x_\alpha)s_\beta(y_\beta)) \\ &= (1 - s_{n-1}d_{n-1})\dots(1 - s_0 d_0)(s_\alpha(x_\alpha)s_\beta(y_\beta)). \end{aligned}$$

We now define the ideal I_n to be that generated by elements of the form

$$C_{\alpha,\beta}(x_\alpha \otimes y_\beta)$$

where $x_\alpha \in NE_{n-\#\alpha}$ and $y_\beta \in NE_{n-\#\beta}$.

We examine this ideal for $n = 2$ and $n = 3$ to see what it looks like.

Example 2.3.7 *For $n = 2$, suppose $\alpha = (1),\ \beta = (0)$ and $x, y \in NE_1 = \mathrm{Ker} d_0$. It follows that*

$$\begin{aligned} C_{(1)(0)}(x \otimes y) &= p_1 p_0 (s_1 x s_0 y) \\ &= s_1 x s_0 y - s_1 x s_1 y \\ &= s_1 x (s_0 y - s_1 y) \end{aligned}$$

which is a generator element of the ideal I_2.

For $n = 3$, the linear morphisms are the following

$$\begin{array}{lll} C_{(1,0)(2)}, & C_{(2,0)(1)}, & C_{(2,1)(0)}, \\ C_{(2)(0)}, & C_{(2)(1)}, & C_{(1)(0)}. \end{array}$$

For all $x \in NE_1,\ y \in NE_2$, the corresponding generators of I_3 are:

$$
\begin{aligned}
C_{(1,0)(2)}(x \otimes y) &= (s_1 s_0 x - s_2 s_0 x) s_2 y, \\
C_{(2,0)(1)}(x \otimes y) &= (s_2 s_0 x - s_2 s_1 x)(s_1 y - s_2 y), \\
C_{(2,1)(0)}(x \otimes y) &= s_2 s_1 x (s_0 y - s_1 y + s_2 y);
\end{aligned}
$$

whilst for all $x,\ y \in NE_2$,

$$
\begin{aligned}
C_{(1)(0)}(x \otimes y) &= s_1 x (s_0 y - s_1 y) + s_2 (xy), \\
C_{(2)(0)}(x \otimes y) &= (s_2 x)(s_0 y), \\
C_{(2)(1)}(x \otimes y) &= s_2 x (s_1 y - s_2 y).
\end{aligned}
$$

In the following we analyse various types of elements in I_n and show that sums of them give elements that we want in giving an alternative description of $\partial_n NE_n$ in certain cases.

Lemma 2.3.8 *Given $x_\alpha \in NE_{n-\#\alpha},\ y_\beta \in NE_{n-\#\beta}$ with $\alpha = (i_r, \ldots, i_1)$, $\beta = (j_s, \ldots, j_1) \in S(n)$. If $\alpha \cap \beta = \emptyset$ with $\alpha < \beta$ and $u = s_\alpha(x_\alpha) s_\beta(y_\beta)$, then*

(i) if $k \leq j_1$, then $p_k(u) = u$,

(ii) if $k > j_s + 1$ or $k > i_r + 1$, then $p_k(u) = u$,

(iii) if $k \in \{i_1, \ldots, i_r, i_r + 1\}$ and $k = j_l + 1$ for some l, then

$$p_k(u) = s_\alpha(x_\alpha) s_\beta(y_\beta) - s_k(z_k),$$

for some $z_k \in E_{n-1}$,

(iv) if $k \in \{j_1, \ldots, j_s, j_s + 1\}$ and $k = i_m + 1$ for some m, then

$$p_k(u) = s_\alpha(x_\alpha) s_\beta(y_\beta) - s_k(z_k),$$

where $z_k \in E_{n-1}$ and $0 \leq k \leq n-1$.

Proof: Assuming $\alpha < \beta$ and $\alpha \cap \beta = \emptyset$ which implies $j_1 < i_1$. In the range $0 \leq k \leq j_1$,

$$
\begin{aligned}
p_k(u) &= s_\alpha(x_\alpha) s_\beta(y_\beta) - (s_k d_k s_\alpha x_\alpha)(s_k d_k s_\beta y_\beta) \\
&= s_\alpha(x_\alpha) s_\beta(y_\beta) - (s_k s_{i_r - 1} \ldots s_{i_1 - 1} d_k x_\alpha)(s_k d_k s_\beta y_\beta) \\
&= s_\alpha(x_\alpha) s_\beta(y_\beta) \qquad \text{since } d_k(x_\alpha) = 0.
\end{aligned}
$$

Similarly if $k > j_s + 1$, then

$$\begin{aligned} p_k(u) &= s_\alpha(x_\alpha)s_\beta(y_\beta) - (s_k d_k s_\alpha x_\alpha)(s_k d_k s_\beta y_\beta) \\ &= s_\alpha(x_\alpha)s_\beta(y_\beta) - (s_k d_k s_\alpha x_\alpha)(s_k s_{j_s} \dots s_{j_1} d_{k-s} y_\beta) \\ &= s_\alpha(x_\alpha)s_\beta(y_\beta) \qquad \text{since } d_{k-s}(y_\beta) = 0. \end{aligned}$$

Clearly the same sort of argument works if $k > i_r + 1$.
If $k \in \{i_1, \dots, i_r, i_r + 1\}$ and $k = j_l + 1$ for some l, then

$$\begin{aligned} p_k(u) &= s_\alpha(x_\alpha)s_\beta(y_\beta) - s_k[d_k(s_\alpha(x_\alpha)s_\beta(y_\beta))] \\ &= s_\alpha(x_\alpha)s_\beta(y_\beta) - s_k(z_k) \end{aligned}$$

where $z_k = s_{\alpha'}(x_{\alpha'})s_{\beta'}(y_{\beta'}) \in E_{n-1}$ for new strings α', β' as is clear. The proof of (iv) is same so we will leave it out. □

Lemma 2.3.9 *If $\alpha \cap \beta = \emptyset$ and $\alpha < \beta$, then*

$$p_{n-1} \dots p_0(s_\alpha(x_\alpha)s_\beta(y_\beta)) = s_\alpha(x_\alpha)s_\beta(y_\beta) - \sum_{k=1}^{n-1} s_k(z_k)$$

where $z_k \in E_{n-1}$.

Proof: We prove this by using the induction hypothesis on n. Write $u = s_\alpha(x_\alpha)s_\beta(y_\beta)$. For $n = 1$, it is clear to see that the equality is verified. We suppose that it is true for $n-2$. It then follows that

$$\begin{aligned} p_{n-1} \dots p_0(u) &= p_{n-1}(u - \sum_{k=1}^{n-2} s_k(z_k)) \\ &= p_{n-1}(u) - p_{n-1}(\sum_{k=1}^{n-2} s_k(z_k)) \end{aligned}$$

as p_{n-1} is a linear map. Next look at $p_{n-1}(u) = u - s_{n-1}(\underbrace{d_{n-1}u}_{z'}) = u - s_{n-1}(z')$ and

$$\begin{aligned} p_{n-1}(\sum_{k=1}^{n-2} s_k(z_k)) &= \sum_{k=1}^{n-2} s_k(z_k) - s_{n-1}(\underbrace{\sum_{k=1}^{n-2} d_{n-1}s_k(z_k)}_{z''}) \\ &= \sum_{k=1}^{n-2} s_k(z_k) - s_{n-1}(z''). \end{aligned}$$

Thus

$$\begin{aligned}
p_{n-1}\dots p_0(u) &= u-\sum_{k=1}^{n-2}s_k(z_k)+s_{n-1}(\underbrace{z''-z'}_{z_{n-1}}) \\
&= u-\sum_{k=1}^{n-2}s_k(z_k)+s_{n-1}(z_{n-1}) \\
&= u-\sum_{k=1}^{n-1}s_k(z_k).
\end{aligned}$$

as required. □

Note that: For $x,y\in NE_{n-1}$, it is easy to see that

$$p_{n-1}\dots p_0(s_{n-1}(x)s_{n-2}(y))=s_{n-1}(x)(s_{n-2}y-s_{n-1}y)$$

and taking the image of this element by d_n gives

$$d_n[s_{n-1}(x)(s_{n-2}y-s_{n-1}y)]=x(s_{n-2}d_{n-1}y-y)$$

which gives a Peiffer type element of order n.

Lemma 2.3.10 *Let $x_\alpha\in NE_{n-\#\alpha}$, $y_\beta\in NE_{n-\#\beta}$ with $\alpha,\beta\in S(n)$, then*

$$s_\alpha(x_\alpha)s_\beta(y_\beta)=s_{\alpha\cap\beta}(z_{\alpha\cap\beta})$$

where $z_{\alpha\cap\beta}$ has the form $(s_{\alpha'}x_\alpha)(s_{\beta'}y_\beta)$ and $\alpha'\cap\beta'=\emptyset$.

Proof: If $\alpha\cap\beta=\emptyset$, then this is trivially true. Assume $\#(\alpha\cap\beta)=t$, with $t\in\mathbb{N}$. Take $\alpha=(i_r,\dots,i_1)$ and $\beta=(j_s,\dots,j_1)$ with $\alpha\cap\beta=(k_t,\dots,k_1)$,

$$s_\alpha(x_\alpha)=s_{i_r}\dots s_{k_t}\dots s_{i_1}(x_\alpha)\quad\text{and}\quad s_\beta(y_\beta)=s_{j_s}\dots s_{k_t}\dots s_{j_1}(y_\beta).$$

Using repeatedly the simplicial axiom $s_es_d=s_ds_{e-1}$ for $d<e$ until obtaining that $s_{k_t}\dots s_{k_1}$ is at beginning of the string, one gets the following

$$s_\alpha(x_\alpha)=s_{k_t\dots k_1}(s_{\alpha'}x_\alpha)\quad\text{and}\quad s_\beta(y_\beta)=s_{k_t\dots k_1}(s_{\beta'}y_\beta).$$

Multiplying these expressions together gives

$$\begin{aligned}
s_\alpha(x_\alpha)s_\beta(y_\beta) &= s_{k_t}\dots s_{k_1}(s_{\alpha'}x_\alpha)s_{k_t}\dots s_{k_1}(s_{\beta'}y_\beta) \\
&= s_{k_t\dots k_1}((s_{\alpha'}x_\alpha)(s_{\beta'}y_\beta)) \\
&= s_{\alpha\cap\beta}(z_{\alpha\cap\beta}),
\end{aligned}$$

where $z_{\alpha\cap\beta} = (s_{\alpha'}x_\alpha)(s_{\beta'}y_\beta) \in E_{n-\#(\alpha\cap\beta)}$ and where $\alpha\backslash\alpha\cap\beta = \alpha'$, $\beta\backslash\alpha\cap\beta = \beta'$. Hence $\alpha'\cap\beta' = \emptyset$. Moreover $\alpha' < \alpha$ and $\beta' < \beta$ as $\#\alpha' < \#\alpha$ and $\#\beta' < \#\beta$. □

Proposition 2.3.11 *Let* **E** *be a simplicial algebra and* $n > 0$, *and* D_n *the ideal in* E_n *generated by degenerate elements. We suppose* $E_n = D_n$, *and let* I_n *be the ideal generated by elements of the form*

$$C_{\alpha,\beta}(x_\alpha \otimes y_\beta) \qquad \text{with } (\alpha,\beta) \in P(n)$$

where $x_\alpha \in NE_{n-\#\alpha}$, $y_\beta \in NE_{n-\#\beta}$. *Then*

$$\partial_n(NE_n) = \partial_n(I_n).$$

Proof: From proposition 2.2.3, E_n is isomorphic to

$$NE_n \rtimes s_{n-1}NE_{n-1} \rtimes s_{n-2}NE_{n-1} \rtimes \ldots \rtimes s_{n-1}s_{n-2}\ldots s_0NE_0,$$

here $NE_n = \bigcap_{i=0}^{n-1}\mathrm{Ker}d_i$ and $NE_0 = E_0$. Hence any element x in E_n can be written in the following form

$$x = e_n + s_{n-1}(x_{n-1}) + s_{n-2}(x'_{n-1}) + s_{n-1}s_{n-2}(x_{n-2}) + \ldots + s_{n-1}s_{n-2}\ldots s_0(x_0),$$

with $e_n \in NE_n$, $x_{n-1}, x'_{n-1} \in NE_{n-1}$, $x_{n-2} \in NE_{n-2}$, $x_0 \in NE_0$ etc.

We start by comparing I_n with NE_n. We show $NE_n = I_n$. It is enough to prove that, equivalently, any element in E_n/I_n can be written

$$s_{n-1}(x_{n-1}) + s_{n-2}(x'_{n-1}) + s_{n-1}s_{n-2}(x_{n-2}) + \ldots + s_{n-1}s_{n-2}\ldots s_0(x_0) + I_n$$

which implies, for any $b \in E_n$,

$$b + I_n = s_{n-1}(x_{n-1}) + s_{n-2}(x'_{n-1}) + \ldots + s_{n-1}s_{n-2}\ldots s_0(x_0) + I_n.$$

for some $x_{n-1} \in NE_{n-1}$ etc.

If $b \in E_n$, it is a sum of products of degeneracies so first of all assume it to be a product of degeneracies and that will suffice for the general case.

If b is itself a degenerate element, it is obvious that it is in some semidirect factor $s_\alpha(E_{n-\#\alpha})$. Assume therefore that provided an element b can be written as a product of

$k-1$ degeneracies it has the desired form mod I_n, now for an element b which needs k degenerate elements

$$b = s_\beta(y_\beta)b' \qquad \text{with } y_\beta \in NE_{n-\#\beta}$$

where b' needs fewer than k and so

$$\begin{aligned} b + I_n &= s_\beta(y_\beta)(b' + I_n) \\ &= s_\beta(y_\beta)(s_{n-1}(x_{n-1}) + s_{n-2}(x'_{n-1}) + \ldots + s_{n-1}s_{n-2}\ldots s_0(x_0) + I_n) \\ &= \sum_{\alpha \in S(n)} s_\beta(y_\beta)s_\alpha(x_\alpha) + I_n. \end{aligned}$$

Next we ignore this summation and just look at the product

$$s_\alpha(x_\alpha)s_\beta(y_\beta) \qquad (*).$$

We check this product case by case as follows:

If $\alpha \cap \beta = \emptyset$, then there exists by lemma 2.3.8 and 2.3.9, an element $s_\alpha(x_\alpha)s_\beta(y_\beta) - \sum_{k=1}^{n-1} s_k(z_k)$ in I_n with $z_k \in E_{n-1}$ and $k \in \alpha$ so that

$$s_\alpha(x_\alpha)s_\beta(y_\beta) \equiv \sum_{k=1}^{n-1} s_k(z_k) \text{ mod } I_n.$$

If $\alpha \cap \beta \neq \emptyset$, then one gets, from lemma 2.3.10, the following

$$s_\alpha(x_\alpha)s_\beta(y_\beta) = s_{\alpha\cap\beta}(z_{\alpha\cap\beta})$$

where $z_{\alpha\cap\beta} = (s_{\alpha'}x_\alpha)(s_{\beta'}y_\beta) \in E_{n-\#(\alpha\cap\beta)}$, with $t \in \mathbb{N}$. Since $\alpha' \cap \beta' = \emptyset$, we can use lemma 2.3.9 to form an equality

$$s_{\alpha'}(x_\alpha)s_{\beta'}(y_\beta) \equiv \sum_{k'=0}^{n-1} s_{k'}(z_{k'}) \quad \text{mod } I_n$$

where $z_{k'} \in E_{n-1}$. It then follows that

$$\begin{aligned} s_{\alpha\cap\beta}(z_{\alpha\cap\beta}) &= s_{\alpha\cap\beta}((s_{\alpha'}x_\alpha)(s_{\beta'}y_\beta)) \\ &\equiv \sum_{k'=0}^{n-1} s_{\alpha\cap\beta}s_{k'}(z_{k'}) \quad \text{mod } I_n. \end{aligned}$$

Thus we have shown that every product which can be formed in the required form are in I_n. Therefore $\partial_n(I_n) = \partial_n(NE_n)$. □

2.4 The cases $n=2$ and $n=3$

2.4.1 Case $n=2$

We know that any element e_2 of E_2 can be expressed in the form

$$e_2 = b + s_1 y + s_0 x + s_0 u$$

with $b \in NE_2, x, y \in NE_1$ and $u \in s_0 E_0$. We suppose $D_2 = E_2$. For $n = 1$, we take $\alpha = (1)$, $\beta = (0)$ and $x, y \in NE_1 = \text{Ker}d_0$. By example 2.3.7, the ideal I_2 is generated by elements of the form

$$C_{(1)(0)}(x \otimes y) = s_1 x(s_0 y - s_1 y).$$

The image of I_2 by ∂_2 is known to be $\text{Ker}d_0 \text{Ker}d_1$ by direct calculation. Indeed,

$$\begin{aligned} d_2[C_{(1)(0)}(x \otimes y)] &= d_2[s_1 x(s_0 y - s_1 y)] \\ &= x(s_0 d_1 y - y) \end{aligned}$$

where $x \in \text{Ker}d_0$ and $(s_0 d_1 y - y) \in \text{Ker}d_1$ and all elements of $\text{Ker}d_1$ have this form by lemma 2.3.1. Thus $\partial_2(I_2) \subseteq \text{Ker}d_0 \text{Ker}d_1 = K_{\{0\}} K_{\{1\}} = K_I K_J$. Using similar calculations to those in example 2.3.5, it is easy to obtain the converse of the equality and so $\partial_2(I_2) = \text{Ker}d_0 \text{Ker}d_1$. We can summarise this in the following table

α	β	I, J
(1)	(0)	$\{0\}$ $\{1\}$

Let us illustrate the product $(*)$ of proposition 2.3.11. For $x', y' \in NE_1$ and $v \in s_0 E_0$, the first case is

$$\begin{aligned} (s_1(y) + s_0(x) + s_0(u)) s_0(v) &= s_1(y) s_0(v) + s_0(xv) + s_0(uv) \\ &= s_1(yv) + s_0(xv) + s_0(uv) \end{aligned}$$

since

$$\begin{aligned} s_1(yv) &= s_1(y) s_1(v) \\ &= s_1(y) s_1(s_0(v')) \quad \text{with } v' \in E_0 \\ &= s_1(y) s_0 s_0(v') \\ &= s_1(y) s_0(v) \end{aligned}$$

It is also easily seen that yv and $xv \in NE_1$ whilst $uv \in s_0 E_0$.

The second case is

$$\begin{aligned}(s_1(y)+s_0(x)+s_0(u))s_0(x') &= s_1(y)s_0(x')+s_0(xx')+s_0(ux')\\ &\equiv s_1(yx')+s_0(xx')+s_0(ux'),\end{aligned}$$

since $s_1(y)(s_0(x')-s_1(x'))\equiv 0 \mod I_2$.

For the third case, we need the identity

$$s_0(x)(s_1(y)-s_0(y))\equiv 0 \mod I_2$$

and so $s_0(x)s_1(y)\equiv s_0(x)s_0(y)$. Hence we have

$$\begin{aligned}(s_0(y)+s_1(x)+s_1(u))s_1(y') &= s_0(y)s_1(y')+s_1(xy')+s_1(uy')\\ &\equiv s_0(yy')+s_1(xy')+s_1(uy')\end{aligned}$$

Hence $\partial_2(I_2)=\partial_2(NE_2)$ as claimed.

2.4.2 Case $n=3$

This subsection provides analogues in dimension 3 of the Peiffer elements.

Proposition 2.4.1

$$\partial_3(NE_3)=\sum_{I,J}K_IK_J+K_{\{0,1\}}K_{\{0,2\}}+K_{\{0,2\}}K_{\{1,2\}}+K_{\{0,1\}}K_{\{1,2\}}$$

where $I\cup J=[2]$, $I\cap J=\emptyset$ *and*

$$\begin{aligned}K_{\{0,1\}}K_{\{0,2\}} &= (\mathrm{Ker}d_0\cap\mathrm{Ker}d_1)(\mathrm{Ker}d_0\cap\mathrm{Ker}d_2)\\ K_{\{0,2\}}K_{\{1,2\}} &= (\mathrm{Ker}d_0\cap\mathrm{Ker}d_2)(\mathrm{Ker}d_1\cap\mathrm{Ker}d_2)\\ K_{\{0,1\}}K_{\{1,2\}} &= (\mathrm{Ker}d_0\cap\mathrm{Ker}d_1)(\mathrm{Ker}d_1\cap\mathrm{Ker}d_2).\end{aligned}$$

Proof: By example 2.3.7 and proposition 2.3.11, we know the generator elements of the ideal I_3 and $\partial_3(I_3)=\partial_3(NE_3)$. The image of all the listed generator elements of the ideal I_3 can be given in the following table.

	α	β	I,J
1	(1,0)	(2)	{2} {0,1}
2	(2,0)	(1)	{1} {0,2}
3	(2,1)	(0)	{0} {1,2}
4	(2)	(1)	{0,1} {0,2}
5	(2)	(0)	{0,1} {1,2}+{0,1} {0,2}
6	(1)	(0)	{0,2} {1,2}+{0,1} {1,2}+{0,1} {0,2}

The explanation of this table is the following:

Row 1.

Firstly we look at the case of $\alpha = (1,0)$ and $\beta = (2)$. For $x \in NE_1$ and $y \in NE_2$,

$$\begin{aligned} d_3[C_{(1,0)(2)}(x \otimes y)] &= d_3[(s_1s_0x - s_2s_0x)s_2y] \\ &= (s_1s_0d_1x - s_0x)y \end{aligned}$$

and so

$$d_3[C_{(1,0)(2)}(x \otimes y)] = (s_1s_0d_1x - s_0x)y \in \mathrm{Ker}d_2(\mathrm{Ker}d_0 \cap \mathrm{Ker}d_1).$$

We have denoted $\mathrm{Ker}d_2(\mathrm{Ker}d_0 \cap \mathrm{Ker}d_1)$ by $K_{\{2\}}K_{\{0,1\}}$ where $I = \{2\}$ and $J = \{0,1\}$.

Row 2. For $\alpha = (2,0),\ \beta = (1)$ with $x \in NE_1,\ y \in NE_2$,

$$\begin{aligned} d_3[C_{(2,0)(1)}(x \otimes y)] &= d_3[(s_2s_0x - s_2s_1x)(s_1y - s_2y)] \\ &= (s_0x - s_1x)(s_1d_2y - y) \end{aligned}$$

and so

$$d_3[C_{(2,0)(1)}(x \otimes y)] \in \mathrm{Ker}d_1(\mathrm{Ker}d_0 \cap \mathrm{Ker}d_2) = K_{\{1\}}K_{\{0,2\}}.$$

Row 3. For $\alpha = (2,1),\ \beta = (0)$ with $x \in NE_1,\ y \in NE_2$,

$$\begin{aligned} d_3[C_{(2,1)(0)}(x \otimes y)] &= d_3[s_2s_1x(s_0y - s_1y + s_2y)] \\ &= s_1x(s_0d_2y - s_1d_2y + y) \end{aligned}$$

and hence

$$d_3[C_{(2,1)(0)}(x \otimes y)] \in \mathrm{Ker}d_0(\mathrm{Ker}d_1 \cap \mathrm{Ker}d_2) = K_{\{0\}}K_{\{1,2\}}.$$

Row 4. For $\alpha=(2),\ \beta=(1)$ with $x,\ y\in NE_2=\mathrm{Ker}d_0\cap\mathrm{Ker}d_1$,

$$\begin{aligned} d_3[C_{(2)(1)}(x\otimes y)] &= d_3[s_2xs_1y-s_2xs_2y] \\ &= x(s_1d_2y-y). \end{aligned}$$

It follows that

$$\begin{aligned} d_3[C_{(2)(1)}(x\otimes y)] &\in (\mathrm{Ker}d_0\cap\mathrm{Ker}d_1)(\mathrm{Ker}d_0\cap\mathrm{Ker}d_2) \\ &= K_{\{0,1\}}K_{\{0,2\}}. \end{aligned}$$

Row 5. For $\alpha=(2),\ \beta=(0)$ with $x,\ y\in NE_2=\mathrm{Ker}d_0\cap\mathrm{Ker}d_1$,

$$\begin{aligned} d_3[C_{(2)(0)}(x\otimes y)] &= d_3[s_2xs_0y] \\ &= xs_0d_2y. \end{aligned}$$

We can assume, for $x,\ y\in NE_2$,

$$x\in\mathrm{Ker}d_0\cap\mathrm{Ker}d_1 \quad\text{and}\quad y+s_0d_2y-s_1d_2y\in\mathrm{Ker}d_1\cap\mathrm{Ker}d_2$$

and, multiplying them together,

$$\begin{aligned} x(y+s_0d_2y-s_1d_2y) &= xy+xs_0d_2y-xs_1d_2y \\ &= x(y-s_1d_2y)+xs_0d_2y \\ &= d_3[C_{(2)(1)}(x\otimes y)]+d_3[C_{(2)(0)}(x\otimes y)] \end{aligned}$$

and so

$$\begin{aligned} d_3[C_{(2)(0)}(x\otimes y)] &\in K_{\{0,1\}}K_{\{1,2\}}+d_3[C_{(2)(1)}(x\otimes y)] \\ &\subseteq K_{\{0,1\}}K_{\{1,2\}}+K_{\{0,1\}}K_{\{0,2\}}. \end{aligned}$$

Row 6. For $\alpha=(1),\ \beta=(0)$ and $x,\ y\in NE_2=\mathrm{Ker}d_0\cap\mathrm{Ker}d_1$,

$$\begin{aligned} d_3[C_{(1)(0)}(x\otimes y)] &= d_3[s_1xs_0y-s_1xs_1y+s_2xs_2y] \\ &= s_1d_2xs_0d_2y-s_1d_2xs_1d_2y+xy. \end{aligned}$$

We can take the following elements

$$(s_0d_2y-s_1d_2y+y)\in\mathrm{Ker}d_1\cap\mathrm{Ker}d_2 \quad\text{and}\quad (s_1d_2x-x)\in\mathrm{Ker}d_0\cap\mathrm{Ker}d_2.$$

When we multiply them together, we get

$$\begin{aligned}(s_0d_2y - s_1d_2y + y)(s_1d_2x - x) &= [s_0d_2ys_1d_2x - s_1d_2ys_1d_2x + yx] \\ &\quad -[xs_0d_2y] + [x(s_1d_2y - y)] \\ &\quad +[y(s_1d_2x - x)] \\ &= d_3[C_{(1)(0)}(x\otimes y)] - d_3[C_{(2)(0)}(x\otimes y)] + \\ &\quad d_3[C_{(2)(1)}(x\otimes y) + C_{(2)(1)}(y\otimes x)]\end{aligned}$$

and hence

$$d_3[C_{(1)(0)}(x\otimes y)] \in K_{\{0,2\}}K_{\{1,2\}} + K_{\{0,1\}}K_{\{1,2\}} + K_{\{0,1\}}K_{\{0,2\}}.$$

So we have shown

$$\partial_3(NE_3) \subseteq \sum_{I,J} K_IK_J + K_{\{0,1\}}K_{\{0,2\}} + K_{\{0,2\}}K_{\{1,2\}} + K_{\{0,1\}}K_{\{1,2\}}.$$

The opposite inclusion can be verified by using proposition 2.3.4. Therefore

$$\begin{aligned}\partial_3(NE_3) &= \mathrm{Ker}d_2(\mathrm{Ker}d_0 \cap \mathrm{Ker}d_1) + \mathrm{Ker}d_1(\mathrm{Ker}d_0 \cap \mathrm{Ker}d_2) + \\ &\quad \mathrm{Ker}d_0(\mathrm{Ker}d_1 \cap \mathrm{Ker}d_2) + (\mathrm{Ker}d_0 \cap \mathrm{Ker}d_1)(\mathrm{Ker}d_0 \cap \mathrm{Ker}d_2) + \\ &\quad (\mathrm{Ker}d_1 \cap \mathrm{Ker}d_2)(\mathrm{Ker}d_0 \cap \mathrm{Ker}d_2) + (\mathrm{Ker}d_1 \cap \mathrm{Ker}d_2)(\mathrm{Ker}d_0 \cap \mathrm{Ker}d_1).\end{aligned}$$

This completes the proof of the proposition. □

2.5 The case $n = 4$

Proposition 2.5.1

$$\partial_4(NE_4) = \sum_{I,J} K_IK_J$$

where $I \cup J = [3],\ I = [3] - \{\alpha\},\ J = [3] - \{\beta\}$ *and* $(\alpha, \beta) \in P(4)$.

Proof: There is a natural isomorphism

$$\begin{aligned}E_4 \cong\ & NE_4 \rtimes s_3NE_3 \rtimes s_2NE_3 \rtimes s_3s_2NE_2 \rtimes s_1NE_3 \rtimes \\ & s_3s_1NE_2 \rtimes s_2s_1NE_2 \rtimes s_3s_2s_1NE_1 \rtimes s_0NE_3 \rtimes \\ & s_3s_0NE_2 \rtimes s_2s_0NE_2 \rtimes s_3s_2s_0NE_1 \rtimes \\ & s_1s_0NE_2 \rtimes s_3s_1s_0NE_1 \rtimes s_3s_2s_1s_0NE_0.\end{aligned}$$

We firstly see what the generator elements of the ideal I_4 look like in the following: For $n = 4$, one gets

$$
\begin{aligned}
S(4) &= \{\emptyset_4 < (3) < (2) < (3,2) < (1) < (3,1) < (2,1) < (3,2,1) < (0) < \\
&\quad (3,0) < (2,0) < (3,2,0) < (1,0) < (3,1,0) < (3,2,1,0)\}.
\end{aligned}
$$

The linear morphisms are the following:

$$
\begin{array}{llll}
C_{(3,2,1)(0)} & C_{(3,2,0)(1)} & C_{(3,1,0)(2)} & C_{(2,1,0)(3)} \\
C_{(3,2)(1,0)} & C_{(3,1)(2,0)} & C_{(3,0)(2,1)} & C_{(3,2)(1)} \\
C_{(3,2)(0)} & C_{(3,1)(2)} & C_{(3,1)(0)} & C_{(3,0)(2)} \\
C_{(3,0)(1)} & C_{(2,1)(3)} & C_{(2,1)(0)} & C_{(2,0)(3)} \\
C_{(2,0)(1)} & C_{(1,0)(3)} & C_{(1,0)(2)} & C_{(3)(2)} \\
C_{(3)(1)} & C_{(3)(0)} & C_{(2)(1)} & C_{(2)(0)} \\
C_{(1)(0)}. & & &
\end{array}
$$

For $x_1, y_1 \in NE_1,\ x_2, y_2 \in NE_2$ and $x_3, y_3 \in NE_3$, the generator elements of the ideal I_4 are

$$
\begin{array}{lrcl}
1) & C_{(3,2,1)(0)}(x_1 \otimes y_3) & = & s_3s_2s_1x_1(s_0y_3 - s_1y_3 + s_2y_3 - s_3y_3) \\
2) & C_{(3,2,0)(1)}(x_1 \otimes y_3) & = & (s_3s_2s_0x_1 - s_1s_2s_1x_1)(s_1y_3 - s_2y_3 + s_3y_3) \\
3) & C_{(3,1,0)(2)}(x_1 \otimes y_3) & = & (s_3s_1s_0x_1 - s_2s_2s_0x_1)(s_2y_3 - s_3y_3) \\
4) & C_{(2,1,0)(3)}(x_1 \otimes y_3) & = & (s_2s_1s_0x_1 - s_3s_1s_0x_1)s_3y_3 \\
5) & C_{(3,2)(1,0)}(x_2 \otimes y_2) & = & (s_1s_0x_2 - s_2s_0x_2 + s_3s_0x_2)s_3s_2y_2 \\
6) & C_{(3,1)(2,0)}(x_2 \otimes y_2) & = & (s_3s_1x_2 - s_3s_0x_2 + s_2s_0x_2 - s_1s_1x_2) \\
 & & & (s_3s_1y_2 - s_3s_2y_2) \\
7) & C_{(3,0)(2,1)}(x_2 \otimes y_2) & = & (s_2s_1x_2 - s_3s_1x_2)(s_3s_0y_2 - s_1s_2y_2 + s_2s_2y_2) \\
8) & C_{(3,2)(1)}(x_2 \otimes y_3) & = & s_3s_2x_2(s_1y_3 - s_2y_3 + s_3y_3) \\
9) & C_{(3,2)(0)}(x_2 \otimes y_3) & = & s_3s_2x_2s_0y_3 \\
10) & C_{(3,1)(2)}(x_2 \otimes y_3) & = & (s_2y_3 - s_3y_3)(s_3s_1x_2 - s_2s_2x_2) \\
11) & C_{(3,1)(0)}(x_2 \otimes y_3) & = & s_3s_1x_2(s_0y_3 - s_1y_3) + s_3s_2x_2(s_2y_3 - s_3y_3) \\
12) & C_{(3,0)(2)}(x_2 \otimes y_3) & = & s_3s_0x_2(s_2y_3 - s_3y_3) \\
13) & C_{(3,0)(1)}(x_2 \otimes y_3) & = & s_1y_3(s_3s_0x_2 - s_1s_2x_2) + s_2s_2x_2(s_2y_3 - s_3y_3) \\
14) & C_{(2,1)(3)}(x_2 \otimes y_3) & = & (s_2s_1x_2 - s_3s_1x_2)s_3y_3 \\
15) & C_{(2,1)(0)}(x_2 \otimes y_3) & = & s_2s_1x_2(s_0y_3 - s_1y_3 + s_2y_3) + s_3s_1x_2s_3y_3 \\
16) & C_{(2,0)(3)}(x_2 \otimes y_3) & = & (s_2s_0x_2 - s_3s_0x_2)s_3y_3 \\
17) & C_{(2,0)(1)}(x_2 \otimes y_3) & = & (s_2s_0x_2 - s_1s_1x_2)(s_1y_3 - s_2y_3) + \\
 & & & (s_3s_1x_2 - s_3s_0x_2)s_3y_3 \\
18) & C_{(1,0)(3)}(x_2 \otimes y_3) & = & s_1s_0x_2s_3y_3 \\
19) & C_{(1,0)(2)}(x_2 \otimes y_3) & = & (s_1s_0x_2 - s_2s_0x_2)s_2y_3 + s_3s_0x_2s_3y_3 \\
20) & C_{(3)(2)}(x_3 \otimes y_3) & = & s_3x_3(s_2y_3 - s_3y_3) \\
21) & C_{(3)(1)}(x_3 \otimes y_3) & = & s_3x_3s_1y_3 \\
22) & C_{(3)(0)}(x_3 \otimes y_3) & = & s_3x_3s_0y_3 \\
23) & C_{(2)(1)}(x_3 \otimes y_3) & = & s_2x_3(s_1y_3 - s_2y_3) + s_3(x_3y_3) \\
24) & C_{(2)(0)}(x_3 \otimes y_3) & = & s_2x_3s_0y_3 \\
25) & C_{(1)(0)}(x_3 \otimes y_3) & = & s_1x_3(s_0y_3 - s_1y_3) + s_2(x_3y_3) - s_3(x_3y_3)
\end{array}
$$

By proposition 2.3.11, we have $\partial_4(NE_4) = \partial_4(I_4)$. We take an image by ∂_4 of each $C_{\alpha\beta}$, where $\alpha, \beta \in P(4)$. We summarise the image of all generator elements, which are listed early on, in the subsequent table.

	α	β	I, J
1	(3,2,1)	(0)	{0}{1,2,3}
2	(3,2,0)	(1)	{1}{0,2,3}
3	(3,1,0)	(2)	{2}{0,1,3}
4	(2,1,0)	(3)	{3}{0,1,2}
5	(3,2)	(1,0)	{0,1}{2,3}
6	(3,1)	(2,0)	{0,2}{1,3}
7	(3,0)	(2,1)	{1,2}{0,3}
8	(3,2)	(1)	{0,1}{0,2,3}
9	(3,2)	(0)	{0,1}{1,2,3}+{0,1}{0,2,3}
10	(3,1)	(2)	{0,2}{0,1,3}
11	(3,1)	(0)	{0,2}{1,2,3}+{0,2}{0,1,3}+{0,1}{1,2,3}+{0,1}{0,2,3}
12	(3,0)	(2)	{1,2}{0,1,3}+{0,2}{0,1,3}
13	(3,0)	(1)	{1,2}{0,2,3}+ {0,1}{0,2,3}+{1,2}{0,1,3}+{0,2}{0,1,3}
14	(2,1)	(3)	{0,3}{0,1,2}
15	(2,1)	(0)	{0,3}{1,2,3}+{0,3}{0,1,2}+{0,2}{1,2,3}+{0,2}{0,1,3}
16	(2,0)	(3)	{1,3}{0,1,2}+{0,3}{0,1,2}
17	(2,0)	(1)	{1,3}{0,2,3}+{0,3}{0,1,2}+{1,3}{0,1,2}+{1,2}{0,2,3}+
			{0,2}{0,1,3}+{1,2}{0,1,3}
18	(1,0)	(3)	{2,3}{0,1,2}+{1,3}{0,1,2}
19	(1,0)	(2)	{2,3}{0,1,3}+{1,2}{0,1,3}+{1,3}{0,1,2}+{2,3}{0,1,2}
20	(3)	(2)	{0,1,2}{0,1,3}
21	(3)	(1)	{0,1,2}{0,2,3}+{0,1,2}{0,1,3}
22	(3)	(0)	{0,1,2}{1,2,3}+{0,1,2}{0,2,3}+{0,1,2}{0,1,3}
23	(2)	(1)	{0,1,3}{0,2,3}+{0,1,2}{1,2,3}+{0,1,2}{0,2,3}+

			{0,1,2}{0,1,3}
24	(2)	(0)	{0,1,3}{1,2,3}+{0,1,3}{0,2,3}+{0,1,2}{1,2,3}+
			{0,1,2}{0,2,3}+{0,1,2}{0,1,3}
25	(1)	(0)	{0,2,3}{1,2,3}+{0,1,3}{1,2,3}+{0,1,3}{0,2,3}+
			{0,1,2}{1,2,3}+{0,1,2}{0,2,3}+{0,1,2}{0,1,3}

We now show how each index in the last column of the above table appears.
From number (1) to (7), we can easily show that for $I \cup J = [3],\ I \cap J = \emptyset$,

$$d_4[C_{\alpha,\beta}(x_\alpha \otimes y_\beta)] \in K_I K_J.$$

The rest of them are the following:

Number: 8

$$\begin{aligned} d_4[C_{(3,2)(1)}(x_2 \otimes y_3)] &= s_2x_2(s_1d_3y_3 - s_2d_3y_3 + y_3) \\ &\in K_{\{0,1\}}K_{\{0,2,3\}}. \end{aligned}$$

Number: 9

$$d_4[C_{(3,2)(0)}(x_2 \otimes y_3)] = s_2x_2s_0d_3y_3..$$

Given

$$s_2(x_2) \in K_{\{0,1\}} \quad \text{and} \quad (s_2d_3y_3 - s_1d_3y_3 + s_0d_3y_3 - y_3) \in K_{\{1,2,3\}}.$$

It then follows that

$$\begin{aligned} d_4[C_{(3,2)(0)}(x_2 \otimes y_3)] &\in K_{\{0,1\}}K_{\{1,2,3\}} + d_4[C_{(3,2)(1)}(x_2 \otimes y_3)] \\ &\subseteq K_{\{0,1\}}K_{\{1,2,3\}} + K_{\{0,1\}}K_{\{0,2,3\}}. \end{aligned}$$

Number: 10

$$\begin{aligned} d_4[C_{(3,1)(2)}(x_2 \otimes y_3)] &= (s_1x_2 - s_2x_2)(s_2d_3y_3 - y_3) \\ &\in K_{\{0,2\}}K_{\{0,1,3\}}. \end{aligned}$$

Number: 11

$$d_4[C_{(3,1)(0)}(x_2 \otimes y_3)] = s_1x_2(s_0d_3y_3 - s_1d_3y_3) + s_2x_2(s_2d_3y_3 - y_3).$$

When considering elements

$$(s_2 d_3 y_3 - s_1 d_3 y_3 + s_0 d_3 y_3 - y_3) \in K_{\{1,2,3\}} \text{ and } (s_1 x_2 - s_2 x_2) \in K_{\{0,2\}}$$

and multiplying them together that implies the following

$$\begin{aligned} d_4[C_{(3,1)(0)}(x_2 \otimes y_3)] &\in K_{\{0,2\}}K_{\{1,2,3\}} - d_4[C_{(3,1)(2)}(x_2 \otimes y_3)] + \\ &\quad d_4[C_{(3,2)(0)}(x_2 \otimes y_3)] - d_4[C_{(3,2)(1)}(x_2 \otimes y_3)] \\ &\subseteq K_{\{0,2\}}K_{\{1,2,3\}} + K_{\{0,2\}}K_{\{0,1,3\}} + \\ &\quad K_{\{0,1\}}K_{\{1,2,3\}} + K_{\{0,1\}}K_{\{0,2,3\}}. \end{aligned}$$

Number: 12

$$d_4[C_{(3,0)(2)}(x_2 \otimes y_3)] = s_0 x_2 (s_2 d_3 y_3 - y_3).$$

When given elements

$$(s_2 d_3 y_3 - y_3) \in K_{\{0,1,3\}} \quad \text{and} \quad (s_0 x_2 - s_1 x_2 + s_2 x_2) \in K_{\{1,2\}},$$

one can obtain

$$\begin{aligned} d_4[C_{(3,0)(2)}(x_2 \otimes y_3)] &\in K_{\{1,2\}}K_{\{0,1,3\}} + d_4[C_{(3,1)(2)}(x_2 \otimes y_3)] \\ &\subseteq K_{\{1,2\}}K_{\{0,1,3\}} + K_{\{0,2\}}K_{\{0,1,3\}}. \end{aligned}$$

Number: 13

$$d_4[C_{(3,0)(1)}(x_2 \otimes y_3)] = (s_0 x_2 - s_1 x_2) s_1 d_3 y_3 + s_2 x_2 (s_2 d_3 y_3 - y_3).$$

Having elements

$$(s_0 x_2 - s_1 x_2 + s_2 x_2) \in K_{\{1,2\}} \quad \text{and} \quad (s_1 d_3 y_3 - s_2 d_3 y_3 + y_3) \in K_{\{0,2,3\}}.$$

Then

$$\begin{aligned} d_4[C_{(3,0)(1)}(x_2 \otimes y_3)] &\in K_{\{1,2\}}K_{\{0,2,3\}} - d_4[C_{(3,2)(1)}(x_2 \otimes y_3)] + \\ &\quad d_4[C_{(3,0)(2)}(x_2 \otimes y_3)] - d_4[C_{(3,1)(2)}(x_2 \otimes y_3)] \\ &\subseteq K_{\{1,2\}}K_{\{0,2,3\}} + K_{\{0,1\}}K_{\{0,2,3\}} + \\ &\quad K_{\{1,2\}}K_{\{0,1,3\}} + K_{\{0,2\}}K_{\{0,1,3\}}. \end{aligned}$$

Number: 14

$$\begin{aligned} d_4[C_{(2,1)(3)}(x_2 \otimes y_3)] &= (s_2 s_1 d_2 x_2 - s_1 x_2) y_3 \\ &\in K_{\{0,3\}}K_{\{0,1,2\}}. \end{aligned}$$

Number: 15

$$d_4[C_{(2,1)(0)}(x_2 \otimes y_3)] = s_2 s_1 d_2 x_2(s_0 d_3 y_3 - s_1 d_3 y_3 + s_2 d_3 y_3) + s_1(x_2) y_3.$$

Take elements

$$(s_2 d_3 y_3 - s_1 d_3 y_3 + s_0 d_3 y_3 - y_3) \in K_{\{1,2,3\}} \text{ and } (s_2 s_1 d_2 x_2 - s_1 x_2) \in K_{\{0,3\}}.$$

It follows that

$$\begin{aligned} d_4[C_{(2,1)(0)}(x_2 \otimes y_3)] &\in K_{\{0,3\}} K_{\{1,2,3\}} + d_4[C_{(2,1)(3)}(x_2 \otimes y_3)] + \\ &\quad d_4[C_{(3,1)(0)}(x_2 \otimes y_3)] + d_4[C_{(3,1)(2)}(x_2 \otimes y_3)] \\ &\subseteq K_{\{0,3\}} K_{\{1,2,3\}} + K_{\{0,3\}} K_{\{0,1,2\}} + \\ &\quad K_{\{0,2\}} K_{\{1,2,3\}} + K_{\{0,2\}} K_{\{0,1,3\}}. \end{aligned}$$

Number: 16

$$d_4[C_{(2,0)(3)}(x_2 \otimes y_3)] = (s_2 s_0 d_2 x_2 - s_0 x_2) y_3.$$

Having elements

$$y_3 \in K_{\{0,1,2\}} \quad \text{and} \quad (s_2 s_0 d_2 x_2 - s_0 x_2 + s_1 x_2 - s_1 s_1 d_2 x_2) \in K_{\{1,3\}};$$

then

$$\begin{aligned} d_4[C_{(2,0)(3)}(x_2 \otimes y_3)] &\in K_{\{1,3\}} K_{\{0,1,2\}} - d_4[C_{(2,1)(3)}(x_2 \otimes y_3)] \\ &\subseteq K_{\{1,3\}} K_{\{0,1,2\}} + K_{\{0,3\}} K_{\{0,1,2\}}. \end{aligned}$$

Number: 17

$$\begin{aligned} d_4[C_{(2,0)(1)}(x_2 \otimes y_3)] &= (s_2 s_0 d_2 x_2 - s_1 s_1 d_2 x_2)(s_1 d_3 y_3 - s_2 d_3 y_3) \\ &\quad + y_3(s_1 x_2 - s_0 x_2). \end{aligned}$$

Take elements

$$(s_2 s_0 d_2 x_2 - s_0 x_2 + s_1 x_2 - s_2 s_1 d_2 x_2) \in K_{\{1,3\}} \text{ and } (s_1 d_3 y_3 - s_2 d_3 y_3 + y_3) \in K_{\{0,2,3\}},$$

$$\begin{aligned} d_4[C_{(2,0)(1)}(x_2 \otimes y_3)] &\in K_{\{1,3\}} K_{\{0,2,3\}} - d_4[C_{(2,1)(3)}(x_2 \otimes y_3)] + \\ &\quad d_4[C_{(2,0)(3)}(x_2 \otimes y_3)] + d_4[C_{(3,0)(1)}(x_2 \otimes y_3)] - \\ &\quad d_4[C_{(3,1)(2)}(x_2 \otimes y_3)] + d_4[C_{(3,0)(2)}(x_2 \otimes y_3)] \\ &\subseteq K_{\{1,3\}} K_{\{0,2,3\}} + K_{\{0,3\}} K_{\{0,1,2\}} + \\ &\quad K_{\{1,3\}} K_{\{0,1,2\}} + K_{\{1,2\}} K_{\{0,2,3\}} + \\ &\quad K_{\{0,2\}} K_{\{0,1,3\}} + K_{\{1,2\}} K_{\{0,1,3\}}. \end{aligned}$$

Number: 18

$$d_4[C_{(1,0)(3)}(x_2 \otimes y_3)] = s_1 s_0 d_2(x_2) y_3.$$

Take elements

$$y_3 \in K_{\{0,1,2\}} \quad \text{and} \quad (s_2 s_0 d_2 x_2 - s_0 x_2 - s_1 s_0 d_0 x_2) \in K_{\{2,3\}}.$$

When multiplying them together

$$\begin{aligned} d_4[C_{(1,0)(3)}(x_2 \otimes y_3)] &\in K_{\{2,3\}} K_{\{0,1,2\}} + d_4[C_{(2,0)(3)}(x_2 \otimes y_3)] \\ &\subseteq K_{\{2,3\}} K_{\{0,1,2\}} + K_{\{1,3\}} K_{\{0,1,2\}}. \end{aligned}$$

Number: 19

$$d_4[C_{(1,0)(2)}(x_2 \otimes y_3)] = (s_1 s_0 d_2 x_2 - s_2 s_0 d_2 x_2) s_2 d_3 y_3 + s_0(x_2) y_3.$$

Having elements

$$(s_1 s_0 d_2 x_2 - s_2 s_0 d_2 x_2 + s_0 x_2) \in K_{\{2,3\}} \quad \text{and} \quad (s_2 d_3 y_3 - y_3) \in K_{\{0,1,3\}}$$

one obtains

$$\begin{aligned} d_4[C_{(1,0)(2)}(x_2 \otimes y_3)] &\in K_{\{2,3\}} K_{\{0,1,3\}} - d_4[C_{(3,0)(2)}(x_2 \otimes y_3)] - \\ &\quad d_4[C_{(2,0)(3)}(x_2 \otimes y_3)] - d_4[C_{(1,0)(3)}(x_2 \otimes y_3)] \\ &\subseteq K_{\{2,3\}} K_{\{0,1,3\}} + K_{\{1,2\}} K_{\{0,1,3\}} + \\ &\quad K_{\{1,3\}} K_{\{0,1,2\}} + K_{\{2,3\}} K_{\{0,1,2\}}. \end{aligned}$$

Number: 20

$$\begin{aligned} d_4[C_{(3)(2)}(x_3 \otimes y_3)] &= x_3(s_2 d_3 y_3 - y_3) \\ &\in (\mathrm{Ker} d_0 \cap \mathrm{Ker} d_1 \cap \mathrm{Ker} d_2)(\mathrm{Ker} d_0 \cap \mathrm{Ker} d_1 \cap \mathrm{Ker} d_3) \\ &= K_{\{0,1,2\}} K_{\{0,1,3\}}. \end{aligned}$$

Number: 21

$$d_4[C_{(3)(1)}(x_3 \otimes y_3)] = x_3 s_1 d_3(y_3).$$

Take elements

$$x_3 \in NE_3 = K_{\{0,1,2\}} \quad \text{and} \quad (s_1 d_3 y_3 - s_2 d_3 y_3 + y_3) \in K_{\{0,2,3\}}.$$

When multiplying them together, one gets

$$\begin{aligned} d_4[C_{(3)(1)}(x_3 \otimes y_3)] &\in d_4[C_{(3)(2)}(x_3 \otimes y_3)] + K_{\{0,1,2\}} K_{\{0,2,3\}} \\ &\subseteq K_{\{0,1,2\}} K_{\{0,2,3\}} + K_{\{0,1,2\}} K_{\{0,1,3\}}. \end{aligned}$$

Number: 22

$$d_4[C_{(3)(0)}(x_3 \otimes y_3)] = x_3 s_0 d_3(y_3).$$

Considering elements

$$x_3 \in K_{\{0,1,2\}} \quad \text{and} \quad (s_2 d_3 y_3 - s_1 d_3 y_3 + s_0 d_3 y_3 - y_3) \in K_{\{1,2,3\}},$$

one can get by following the previous steps of the argument

$$\begin{aligned} d_4[C_{(3)(0)}(x_3 \otimes y_3)] &\in K_{\{0,1,2\}}K_{\{1,2,3\}} + d_4[C_{(3)(1)}(x_3 \otimes y_3)] - \\ &\quad d_4[C_{(3)(2)}(x_3 \otimes y_3)] \\ &\subseteq K_{\{0,1,2\}}K_{\{1,2,3\}} + K_{\{0,1,2\}}K_{\{0,2,3\}} + \\ &\quad K_{\{0,1,2\}}K_{\{0,1,3\}}. \end{aligned}$$

Number: 23

$$d_4[C_{(2)(1)}(x_3 \otimes y_3)] = s_2 d_3 x_3 (s_1 d_3 y_3 - s_2 d_3 y_3) + x_3 y_3.$$

Take elements

$$(s_1 d_3 y_3 - s_2 d_3 y_3 + y_3) \in K_{\{0,2,3\}} \quad \text{and} \quad (s_2 d_3 x_3 - x_3) \in K_{\{0,1,3\}}.$$

When putting them together, we obtain

$$\begin{aligned} d_4[C_{(2)(1)}(x_3 \otimes y_3)] &\in K_{\{0,1,3\}}K_{\{0,2,3\}} + d_4[C_{(3)(1)}(x_3 \otimes y_3)] - \\ &\quad d_4[C_{(3)(2)}(x_3 \otimes y_3) + C_{(3)(2)}(y_3 \otimes x_3)] \\ &\subseteq K_{\{0,1,3\}}K_{\{0,2,3\}} + K_{\{0,1,2\}}K_{\{0,2,3\}} + \\ &\quad K_{\{0,1,2\}}K_{\{0,1,3\}}. \end{aligned}$$

Number: 24

$$d_4[C_{(2)(0)}(x_3 \otimes y_3)] = s_2 d_3(x_3) s_0 d_3(y_3).$$

Elements

$$(s_2 d_3 x_3 - x_3) \in K_{\{0,1,3\}} \quad \text{and} \quad (s_2 d_3 y_3 - s_1 d_3 y_3 + s_0 d_3 y_3 - y_3) \in K_{\{1,2,3\}}.$$

It follows that

$$\begin{aligned} d_4[C_{(2)(0)}(x_3 \otimes y_3)] &\in K_{\{0,1,3\}}K_{\{1,2,3\}} - d_4[C_{(2)(1)}(x_3 \otimes y_3)] + \\ & d_4[C_{(3)(0)}(x_3 \otimes y_3)] + d_4[C_{(3)(1)}(x_3 \otimes y_3)] + \\ & d_4[C_{(3)(2)}(x_3 \otimes y_3) + C_{(3)(2)}(y_3 \otimes x_3)] \\ &\subseteq K_{\{0,1,3\}}K_{\{1,2,3\}} + K_{\{0,1,3\}}K_{\{0,2,3\}} + \\ & K_{\{0,1,2\}}K_{\{1,2,3\}} + K_{\{0,1,2\}}K_{\{0,2,3\}} + \\ & K_{\{0,1,2\}}K_{\{0,1,3\}}. \end{aligned}$$

Number: 25

$$d_4[C_{(1)(0)}(x_3 \otimes y_3)] = s_1 d_3(x_3)(s_0 d_3 y_3 - s_1 d_3 y_3) + s_2 d_3(x_3 y_3) - x_3 y_3,$$

and

$$(s_1 d_3 x_3 - s_2 d_3 x_3 + x_3) \in K_{\{0,2,3\}} \text{ and } (s_2 d_3 y_3 - s_1 d_3 y_3 + s_0 d_3 y_3 - y_3) \in K_{\{1,2,3\}},$$

then one can have

$$\begin{aligned} d_4[C_{(1)(0)}(x_3 \otimes y_3)] &\in d_4[C_{(3)(1)}(y_3 \otimes x_3) + C_{(3)(1)}(x_3 \otimes y_3)] - \\ & d_4[C_{(3)(0)}(x_3 \otimes y_3)] + d_4[C_{(2)(0)}(x_3 \otimes y_3)] - \\ & d_4[C_{(2)(1)}(x_3 \otimes y_3) + C_{(2)(1)}(y_3 \otimes x_3)] - \\ & d_4[C_{(3)(2)}(x_3 \otimes y_3) + C_{(3)(2)}(y_3 \otimes x_3)] + \\ & K_{\{0,2,3\}}K_{\{1,2,3\}} \\ &\subseteq K_{\{0,1,2\}}K_{\{0,1,3\}} + \ldots + K_{\{0,2,3\}}K_{\{1,2,3\}}. \end{aligned}$$

So we have shown that for each $C_{\alpha\beta}$, $\partial_4(I_4) \subseteq \sum_{I,J} K_I K_J$. The opposite inclusion of this can be obtained by considering proposition 2.3.4. □

So far we have shown, for $n = 2, 3, 4$, what the image of the Moore complex of a simplicial algebra looks like and also have proved proposition 2.3.4 which is

$$\sum_{I,J} K_I K_J \subseteq \partial_n(NE_n).$$

With respect to all this information, we can identify the following theorem:

Theorem 2.5.2 *Let* $n = 2, 3,$ *or* 4 *and let* **E** *be a simplicial algebra with Moore complex* **NE** *in which* $E_n = D_n$*, Then*

$$\partial_n(NE_n) = \sum_{I,J} K_I K_J$$

for any $I, J \subseteq [n-1]$ *with* $I \cup J = [n-1]$, $I = [n-1] - \{\alpha\}$ *and* $J = [n-1] - \{\beta\}$*, where* $(\alpha, \beta) \in P(n)$.

Theorem 2.5.3 *If* $E_n \neq D_n$*, then*

$$\partial_n(NE_n \cap D_n) = \sum_{I,J} K_I K_J \quad \text{with } n = 2, 3, 4.$$

CHAPTER 3

SIMPLICIAL ALGEBRAS AND CROSSED COMPLEXES

3.1 CROSSED COMPLEXES

The definition of a crossed complex (over a groupoid) was earlier given by R.Brown and P.J.Higgins (1981) [8] generalising earlier work of Whitehead (1949) [43]. The analogue for algebras of the crossed complex is defined by T.Porter (1987) [37]. S.Lichtenbaum and M.Schlessinger [31] and others had considered related ideas in 1967.

Definition 3.1.1 A crossed complex of $\mathbf{k}$-algebras *is a sequence of* $\mathbf{k}$-*algebras*

$$\mathscr{C}: \qquad \cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} R$$

in which

i) ∂_1 *is a crossed R-module,*

ii) for $i > 1$, C_i *is an R-module on which* $\partial_1 C_1$ *operates trivially and each* ∂_i *is an R-module morphism,*

iii) for $i \geq 1, \partial_{i+1}\partial_i = 0$.

Morphisms of crossed complexes are defined in the obvious way : By *a morphism of crossed complexes* $\psi : \mathscr{C} \to \mathscr{D}$, one means a sequence $\psi = \{\psi_n\}$ of algebra morphisms with component $\psi_n : C_n \to D_n$ in degree n, such that the diagrams

$$\begin{array}{ccc} C_n & \xrightarrow{\psi_n} & D_n \\ {\scriptstyle d_n}\downarrow & & \downarrow{\scriptstyle d'_n} \\ C_{n-1} & \xrightarrow[\psi_{n-1}]{} & D_{n-1} \end{array}$$

are commutative for all n and $\psi_n(c_0 \cdot c_n) = \psi_0(c_0) \cdot \psi_n(c_n)$ for all $c_0 \in C_0,\ c_n \in C_n$, where the d'_n denote differentials of the complex $\mathscr{D}$. We therefore get a category of crossed complexes of **k**-algebras denoted by **XComp**. Morphisms ψ_n, given above, are called ψ_0-*equivariant.*

We let **ChComp** denote the category of connected positive chain complexes of modules over **k**-algebras. Thus an object of **ChComp** is a pair $(\underline{\mathscr{C}}, R)$ where R is a **k**-algebra and $\underline{\mathscr{C}}$ is a chain complex of R-modules such that the C_i, $i \leq 0$, are all zero and $d_1 : C_1 \to C_0$ is onto.

Example 3.1.2 *Given* $(\underline{\mathscr{C}}, R)$*, we form a crossed complex,* $\Gamma(\underline{\mathscr{C}}, R)$

$$\cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} R \ltimes C_0.$$

Put $\partial_i = d_i$ *if* $i > 1$, $\partial_1 : C_1 \to R \ltimes C_0$ *given by* $\partial_1(c) = (0, d_1(c))$. *Giving* C_1 *the zero multiplication and a* $R \ltimes C_0$*-module structure via the projection from* $R \ltimes C_0$ *onto* R*, the action of* $R \ltimes C_0$ *on* C_1 *can be given by* $(r, c_0) \cdot c_1 = rc_1$*, that is we make* $R \ltimes C_0$ *act on the* C_i *via the projection onto* R*.*

i) $(C_1, R \ltimes C_0, \partial_1)$ *is a crossed module. For*

$$\begin{aligned} \partial_1(c_1) \cdot c'_1 &= (0, d_1 c_1) \cdot c'_1, \\ &= 0c'_1 = 0, \\ &= c_1 c'_1. \end{aligned}$$

ii) For $i > 1$*, by assumption,* C_i *are* R*-modules. Since* $R \ltimes C_0$ *acts on the* C_i *via the projection onto* R*,* $\partial_1 C_1$ *operates trivially on the* C_i*. Clearly all the* ∂_i *are* R*-module morphisms.*

iii) $\partial_{i+1}\partial_i(c) = 0$ *for all* $i \geq 1$ *from assumption.*

We also say that a crossed complex $\mathscr{C}$ is a *free* if R is a **k**-algebra, C_1 is a free crossed R-module on some function (see Chapter 1) and for $n \geq 2$, C_n is a free R-module on some set X. *The homology of a crossed complex* $\mathscr{C}$ can be defined by

$$H_n(\mathscr{C}) = \mathrm{Ker}\partial_n / \mathrm{Im}\partial_{n+1}.$$

Definition 3.1.3 *A crossed complex $\mathscr{C}$ of k-algebras is* exact *if for $n \geq 1$,*

$$\mathrm{Ker}(\partial_n : C_n \to C_{n-1}) = \mathrm{Im}(\partial_{n+1} : C_{n+1} \to C_n).$$

Definition 3.1.4 A crossed resolution of a k-algebra B *is a crossed complex*

$$\mathscr{C} : \qquad \cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

of **k**-*algebras, where ∂_1 is a crossed C_0-module together with $f : C_0 \to B$ a morphism, such that the sequence*

$$\cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{f} B \to 0$$

is exact.

If, for $i \geq 0$, the C_i are free and ∂_1 a free crossed module, then the resolution is called a free crossed resolution *of the* **k**-*algebra B.*

3.2 Hypercrossed Complexes

Various generalisations of the Dold-Kan theorem (which is an equivalence between the category of simplicial abelian groups and that of positive (abelian) chain complexes) are known. For instance Ashley [3] proves an equivalence between the category of simplicial T-complexes and that of crossed complexes.

P.Carrasco [12] (see also P.Carrasco and A.M.Cegarra [13]) also give the most general non-abelian form of a Dold-Kan type theorem. They show how the Moore complex functor defines a full equivalence between the category of simplicial groups and the category of what are called hypercrossed complexes of groups, i.e. chain complexes of (non-abelian) groups with an extra structure.

Recall the maps, from section 2.3, that

$$C^n_{\alpha,\beta} : NE_{n-\#\alpha} \otimes NE_{n-\#\beta} \longrightarrow NE_n \quad \text{with } (\alpha,\beta) \in P(n)$$

given by

$$C^n_{\alpha,\beta}(x_\alpha \otimes y_\beta) = p(s_\alpha(x_\alpha)s_\beta(y_\beta))$$

for $x_\alpha \in NE_{n-\#\alpha}$ and $y_\beta \in NE_{n-\#\beta}$.

The maps involved in the definition of a hypercrossed complex (see P.Carrasco's thesis [12]). That thesis consists of a proof of the non-abelian Dold-Kan theorem (2.2.9, p.64) presenting an equivalence between the category of simplicial algebras and that of hypercrossed complexes. Another result from [12] is that the category of hypercrossed complexes together with $C^n_{\alpha,\beta} = 0$ is equivalent to that of crossed complexes of algebras.

3.3 From Simplicial Algebras to Crossed Complexes

P.Carrasco and A.M.Cegarra [13] denoted for a simplicial group $\mathbf{G}$,

$$C_n(\mathbf{G}) = \frac{NG_n}{(NG_n \cap D_n)d_{n+1}(NG_{n+1} \cap D_{n+1})}$$

this gives a crossed complex $\mathscr{C}$ from the Moore complex $(\mathbf{NG}, \partial)$. Their proof requires an understanding of hypercrossed complexes. P.J.Ehler and T.Porter [17] developed a direct proof for simplicial groups/groupoids independently of [13]. Here we will do an analogous argument for the algebra case and show that $\mathscr{C}$ is a crossed complex where we shall write

$$C_n(\mathbf{E}) = \frac{NE_n}{(NE_n \cap D_n) + d_{n+1}(NE_{n+1} \cap D_{n+1})}$$

and if $x \in NE_n$, we will write $\bar{x}$ for the corresponding element of $C_n(\mathbf{E})$. The map $\partial_n : C_n(\mathbf{E}) \to C_{n-1}(\mathbf{E})$ will be induced by d_n^n. We denote the nth term of crossed complex $\mathscr{C}$ by $C_n(\mathbf{E})$ instead of C_n as used in [17].

Lemma 3.3.1 *The subalgebra $(NE_n \cap D_n) + d_{n+1}(NE_{n+1} \cap D_{n+1})$ is an ideal in E_n.*

Proof: For any $a \in NE_n \cap D_n, x \in NE_{n+1} \cap D_{n+1}$ and $z \in E_n$, the element $z(a + d_{n+1}x)$ can be written in the following form

$$z(a + d_{n+1}x) = s_{n-1}d_n(z)a + d_{n+1}(s_n(z)x + s_n z s_n a - s_{n-1} z s_n a)$$

and so is in $(NE_n \cap D_n) + d_{n+1}(NE_{n+1} \cap D_{n+1})$. □

By defining

$$\partial_n(\bar{z}) = \overline{d_n^n(z)} \quad \text{with} \quad z \in NE_n,$$

one obtains a well defined map $\partial : C_n(\mathbf{E}) \to C_{n-1}(\mathbf{E})$ verifying $\partial\partial = 0$.

Lemma 3.3.2 *Let $x, y \in E_n$, for $n \geq 2$, then $xy = a + d_{n+1}w$, where*

$$a = (s_{n-2}d_n y - s_{n-1}d_n y)s_{n-1}d_n x \text{ and } w = (s_{n-1}y - s_{n-2}y)s_{n-1}x + s_n x s_n y.$$

Proof: This is immediate by direct calculation. □

Corollary 3.3.3 *If, for $n \geq 2$ $x \in NE_{n-1}$ and $y \in NE_n$, then*

$$s_{n-1}(x)y \in [NE_n \cap D_n + d_{n+1}(NE_{n+1} \cap D_{n+1})].$$

Proof: Replacing x by $s_{n-1}(x)$ in the elements a, w of the previous lemma implies

$$a = s_{n-1}x(s_{n-2}d_n y - s_{n-1}d_n y) \text{ and } w = s_n s_{n-1}x(s_{n-1}y - s_{n-2}y) + (s_n s_{n-1}x)s_n y.$$

It is easy to see that for $i \geq 1$, $d_i a = 0$. Similarly $d_i w = 0$ for all $i \geq 0$. By the previous lemma, the element $s_{n-1}(x)y$ is in $[NE_n \cap D_n + d_{n+1}(NE_{n+1} \cap D_{n+1})]$. □

The significance of this corollary is that the actions of NE_r on NE_n are by multiplication via degeneracies.

In particular we choose the action

$$\bar{x} \cdot \bar{y} = \overline{s_r^{(n-r)}(x)y}$$

where the $(n-r)$-superfix denotes an iterated application of the map. Thus if $n \geq 2$, NE_{n-1} acts trivially on NE_n, as $\overline{s_{n-1}(x)y} = 0$. To satisfy the axioms of a crossed complex, we need to check that C_0 acts on C_n, for $n \geq 1$ and $\partial_1 C_1$ acts trivially on C_n, for $n \geq 2$. To do this, we will give the following lemmas.

Lemma 3.3.4 *For each n, $\partial_n : C_n(\mathbf{E}) \longrightarrow E_{n-1}$ is a crossed module.*

Proof: CM1) For $e \in E_{n-1}, x \in NE_n$, it is clear that since $d_n(s_{n-1}(e)x) = ed_n(x)$, one gets the following

$$\begin{aligned} \partial(e \cdot \bar{x}) &= \partial(s_{n-1}(e)\bar{x}) \\ &= e\partial(\bar{x}). \end{aligned}$$

CM2) We firstly take the element $s_{n-1}d_n(x)y$ with $x \in E_n, y \in NE_n$:

$$s_{n-1}d_n(x)y = xy + d_{n+1}[(s_{n-1}x - s_n x)s_n y]$$

and so

$$s_{n-1}d_n(x)y \equiv xy \mod [NE_n \cap D_n + d_{n+1}(NE_{n+1} \cap D_{n+1})]$$

and for $\bar{x}, \bar{y} \in C_n(\mathbf{E})$

$$\partial \bar{x} \cdot \bar{y} = \overline{s_{n-1}d_n(x)y} \equiv \overline{xy}.$$

This is the verification of the Peiffer identity. □

Later on we use this lemma for $n = 1$.

Lemma 3.3.5 *If $x \in E_{n-i+1}$ and $y \in NE_n$, then for any k, $1 \leq k < i$,*

$$s_n^{(k)} s_{n-i}^{(i-k-1)}(x)y \equiv s_n^{(k-1)} s_{n-i}^{(i-k)}(x)y \mod [NE_n \cap D_n + d_{n+1}(NE_{n+1} \cap D_{n+1})].$$

Proof: Writing $s_n^{(k)}$ for $\underbrace{s_n \dots s_n}_{k-times}$, we consider the element

$$s_n^{(k)} s_{n-i}^{(i-k)}(x) s_k(y) \in E_{n+1}.$$

We recall from section 2.3 that the linear morphisms

$$p_l : E_{n+1} \longrightarrow NE_{n+1} \subset E_{n+1} \quad \text{with } 0 \leq l \leq n$$

given by

$$p_l(z) = z - s_l d_l(z).$$

We also note the particular case of $C_{\alpha,\beta}$, for $\alpha = (n, n-i)$, $\beta = (k)$,

$$\begin{aligned} C_{\alpha,\beta}(x,y) &= C_{(n,n-i),(k)}(x,y) \\ &= p_n \dots p_0(s_n^{(k)} s_{n-i}^{(i-k)}(x) s_k(y)) \in NE_{n+1} \cap D_{n+1}. \end{aligned}$$

We will prove that

$$d_{n+1}(C_{(n,n-i),(k)}(x,y))$$

is basically the difference between the two elements of this lemma.

Indeed, by putting $z_{k,i}(x,y) = s_n^{(k)} s_{n-i}^{(i-k)}(x) s_k(y)$ and recalling lemma 2.3.9, for $\alpha = (n, n-i)$ and $\beta = (k)$ with any j, $0 \leq j \leq n+1$, we obtain

$$d_j z_{k,i}(x,y) = \begin{cases} 0 & \text{if } k > j \\ s_{n-1}^{(k)} s_{n-i}^{(i-k-1)}(x)y & \text{if } k = j \\ s_n^{(k-1)} s_{n-i}^{(i-k)}(x)y & \text{if } k = j-1 \\ 0 & \text{if } 1 < j-i-k+1 \\ 0 & \text{if } j > i+1 \end{cases}$$

and $d_{n+1} z_{k,i}(x,y) = z_{k-1,i-1}(x, d_{n+1}y)$. This gives

$$p_n \dots p_0 z_{k,i}(x,y) = p_n \dots p_{i+k} z_{k,i}(x,y)$$

since the operators p_l for $l > i+1$ are trivial. We also note that

$$p_n \dots p_{i+k} z_{k,i}(x,y) = p_n \dots p_{k+1} z_{k,i}(x,y).$$

Now if $v \in E_{n+1}$, then

$$\begin{array}{rcl} d_{n+1}p_n(v) & = & d_{n+1}v - d_n v \\ d_{n+1}p_n p_{n-1}(v) & = & d_{n+1}p_{n-1}(v) - d_n p_{n-1}(v) \end{array} \qquad (*)$$

and so on. It follows that

$$d_{n+1}p_n \ldots p_{k+1}(z_{k,i}(x,y)) = p_n \ldots p_k(z_{k-1,i-1}(x,d_n y)) - d_n p_{n-1} \ldots p_{k+1}(z_{k,i}(x,y)).$$

The first of these two terms is in $NE_n \cap D_n$ and hence we only check the second one. From $(*)$, we get

$$d_n p_{n-1} \ldots p_{k+1}(v) = d_n p_{n-2} \ldots p_{k+1}(v) - d_{n-1}p_{n-2} \ldots p_{k+1}(v)$$

and this implies

$$d_l p_{l+1} \ldots p_{k+1}(z_{k,i}(x,y))$$

and others of the form

$$d_{l-1}p_{l+1} \ldots p_{k+1}(z_{k,i}(x,y)).$$

If $j < k-1$,

$$d_j p_k(z) = d_j(z) - s_{k-1}d_{k-1}d_j(z) = p_{k-1}d_j(z),$$

so any term of the form $d_{l-1}p_{l+1} \ldots p_{k+1}(z_{k,i}(x,y))$ can be written

$$p_l \ldots p_k(d_{l-1}(z_{k,i}(x,y)))$$

and so is trivial if $l > 1$. Hence the only term is $d_k p_k(z_{k,i}(x,y))$ and so

$$\begin{array}{rcl} d_k p_k(z_{k,i}(x,y)) & = & d_k[s_n^{(k)}s_{n-i}^{(i-k)}(x)s_k(y)] - d_k s_k d_k[s_n^{(k)}s_{n-i}^{(i-k)}(x)s_k(y)] \\ & = & s_n^{(k-1)}s_{n-i}^{(i-k)}(x)y - s_n^{(k)}s_{n-i}^{(i-k-1)}(x)y, \end{array}$$

i.e. the difference of the two terms in the statement of the lemma. Putting

$$t = s_n^{(k-1)}s_{n-i}^{(i-k)}(x)y - s_n^{(k)}s_{n-i}^{(i-k-1)}(x)y.$$

It then follows that

$$d_{n+1}(C_{(n,n-i),(k)}(x,y)) = p_n \ldots p_k(z_{k-1,i-1}(x,d_n y)) - t.$$

Having $p_n \ldots p_k(z_{k-1,i-1}(x,d_n y)) \in NE_n \cap D_n$ and $u \in NE_{n+1} \cap D_{n+1}$ implies that

$$s_n^{(k)}s_{n-i}^{(i-k-1)}(x)y \equiv s_n^{(k-1)}s_{n-i}^{(i-k)}(x)y \text{ mod } [NE_n \cap D_n + d_{n+1}(NE_{n+1} \cap D_{n+1})].$$

This completes the proof . □

The reason for proving this lemma is to give the subsequent one

Lemma 3.3.6 *If $n \geq 1$, $x \in E_{n-i}$ and $y \in NE_n$, then*

$$s_{n-i}^{(i+1)}(d_{n-i}x)y \equiv s_{n-i}^{(i)}(x)y \mod [NE_n \cap D_n + d_{n+1}(NE_{n+1} \cap D_{n+1})].$$

Proof: Take the element $v = s_{n-i}^{(i+1)}(x)s_n(y) - s_n s_{n-i}^{(i)}(x)s_n(y)$. This is $p_n \ldots p_0 s_{n-i}^{(i+1)}(x)s_n(y)$. It is readily checked that $d_i(v) = 0$ for $i \geq 0$ and $d_{n+1}(v)$ is the difference between the elements mentioned in the statement of the lemma. □

Lemma 3.3.7 *If $n \geq 2$, $x \in NE_1$ and $y \in NE_n$, then*

$$s_{n-1} \ldots s_1(x)y \equiv 0 \mod [NE_n \cap D_n + d_{n+1}(NE_{n+1} \cap D_{n+1})].$$

Proof: Consider

$$u = s_n(y)s_n \ldots s_1(x) - s_{n-1}(y)s_n \ldots s_1(x) + \sum_{i=2}^{n}(-1)^i s_{n-i}(y)s_n \ldots s_1(x),$$

and it is easily checked to be in $NE_{n+1} \cap D_{n+1}$. Calculating $d_{n+1}(u)$ gives two terms, i.e.

$$\begin{aligned} d_{n+1}u &= [ys_{n-1} \ldots s_1(x)] - [s_{n-1}d_n(y)s_{n-1} \ldots s_1(x) - \\ &\quad s_{n-2}d_n(y)s_{n-1} \ldots s_1(x) + \\ &\quad \textstyle\sum_{i=2}^{n}(-1)^i s_{n-i}d_n(y)s_{n-1} \ldots s_1(x)]. \end{aligned}$$

Writing

$$\begin{aligned} v &= s_{n-1}d_n(y)s_{n-1} \ldots s_1(x) - s_{n-2}d_n(y)s_{n-1} \ldots s_1(x) + \\ &\quad \textstyle\sum_{i=2}^{n}(-1)^i s_{n-i}d_n(y)s_{n-1} \ldots s_1(x), \end{aligned}$$

it is readily checked that $v \in NE_n$ and is as required. □

The following is originally due to P.Carrasco and A.M.Cegarra [12] for the group case and for the groupoid case due to P.J.Ehler and T.Porter [17].

Proposition 3.3.8 *The construction: for each $n \geq 0$, then setting*

$$C_n(\mathbf{E}) = \frac{NE_n}{(NE_n \cap D_n) + d_{n+1}(NE_{n+1} \cap D_{n+1})} \quad \text{with } \overline{\partial(x)} = \overline{d_n(x)}$$

gives a crossed complex.

Proof: i) from Lemma 3.3.4, $\partial_1 : C_1(\mathbf{E}) \to C_0(\mathbf{E})$ is a crossed module,

ii) Lemma 3.3.2 and Corollary 3.3.3 show that C_0 acts on C_n for $n \geq 1$ via $s_{n-1} \ldots s_0$ and also make C_1 act on $C_n, n \geq 1$ by multiplication via $s_{n-1} \ldots s_1$. Lemma 3.3.6 and repeated use of Lemma 3.3.5 show that if $\overline{x} \in C_1$ then $\overline{x}$ and $\partial_1 \overline{x}$ act on C_n in the same way, and Lemma 3.3.7 gives that $\partial_1 C_1$ acts trivially on C_n,

iii) we noted $\partial\partial = 0$ after Lemma 3.3.1. □

We thus have a functor

$$\mathbf{C} : \mathbf{SimpAlg} \longrightarrow \mathbf{XComp}$$

Remark 3.3.9 $NE_1 \cap D_1 = 0$. Indeed, any element of D_1 has the form $s_0(x)$ for $x \in E_0$, and so if $y \in NE_1 \cap D_1$, then $y = s_0(x)$ for some $x \in E_0$. It follows from $y \in NE_1 = \mathrm{Ker} d_0$ that

$$0 = d_0(y) = d_0 s_0(x) = x$$

which implies $x = 0$ and so $y = 0$ as required. Hence $C_0(\mathbf{E}) = NE_0 = E_0$.

3.4 The Particular Case of a 'Step-by-Step' Construction of a Free Simplicial Algebra and its Skeleton

In this section, we describe the special case of the 'step-by-step' construction of the free simplicial algebra and its skeleton up to dimension 2 and will interpret this construction and see how that relates to other algebraic constructions such as that of a free crossed module, Koszul complexes, and so on.

Let A be a subring of a commutative ring S, and consider the polynomial ring $A[X_1, \ldots, X_n]$ over A in n indeterminates $X_1, \ldots, X_n$. Let $a_1, \ldots, a_n \in S$. There is exactly one ring homomorphism $g : A[X_1, \ldots, X_n] \to S$ with the properties that

$$g(r) = r \qquad \text{for all } r \in A$$

and

$$g(X_i) = a_i \qquad \text{for all } i = 1, \ldots, n.$$

This homomorphism g is called the *evaluation homomorphism* or just evaluation at $a_1, \ldots, a_n$.

If $g : A[X_1, \ldots, X_n] \to A$ is the evaluation homomorphism at $a_1, \ldots, a_n$, then

$$\mathrm{Ker} g = (X_1 - a_1, \ldots, X_n - a_n).$$

For this, see for instance R.Y.Sharp [41].

Let $\mathbf{k}$ be a commutative ring with unit and R be a commutative $\mathbf{k}$-algebra with an ideal $I = (x_1, \ldots, x_n)$ of R generated by the elements $x_1, \ldots, x_n$ in R. Let $\mathbf{K}(R, 0)$ denote the simplicial algebra which in every dimension is equal to R and $d_i = \mathrm{id} = s_j$, for all i, j.

There is an obvious epimorphism:

$$f : R \longrightarrow R/(x_1, \ldots, x_n)$$

which gives an isomorphism $R/\mathrm{Ker}f \cong B$, where $B = R/I$.

Let

$$\Omega^0 = \{x_1, \ldots, x_n\} \subset \mathrm{Ker}f.$$

The 1-skeleton $\mathbf{E}^{(1)}$ of the free simplicial resolution of B can be built by adding new indeterminates $X = \{X_1 \ldots, X_n\}$ into $E_1^{(0)} = R$ to form

$$E_1^{(1)} = E_1^{(0)}[X] = R[X_1, \ldots, X_n],$$

with the face maps and degeneracy map

$$R[X_1, ..., X_n] \underset{s_0}{\overset{d_0, d_1}{\rightrightarrows}} R$$

given by

$$d_1^1(X_i) = x_i \in \mathrm{Ker}f, \quad d_0^1(X_i) = 0, \quad s_0(r) = r \in R.$$

Thus the 1-skeleton $\mathbf{E}^{(1)}$ looks like:

$$\mathbf{E}^{(1)} : \ldots R[s_0(X), s_1(X)] \underset{s_0, s_1}{\overset{d_0, d_1, d_2}{\rightrightarrows}} R[X] \underset{s_0}{\overset{d_0, d_1}{\rightrightarrows}} R \xrightarrow{\ f\ } R/I.$$

Note that for $n > 1$, higher levels of $\mathbf{E}^{(1)}$ are generated by the degenerate elements, so we can apply our results from chapter 2.

Lemma 3.4.1 *We assume given the 1-skeleton $\mathbf{E}^{(1)}$. Let d_0^1 and d_1^1 be evaluation homomorphisms. Then*

i) $\mathrm{Ker}d_0^1 = R^+[X_1, \ldots, X_n] = (X_1, \ldots, X_n)$,

ii) $\mathrm{Ker}d_1^1 = (X_1 - x_1, \ldots, X_n - x_n)$.

Proof: These follows immediately from $\mathrm{Ker}g = (X_1 - a_1, \ldots, X_n - a_n)$, because d_0^1 and d_1^1 are evaluation homomorphisms at $0, \ldots, 0$ and $x_1, \ldots, x_n$ respectively. □

Note $\pi_0(\mathbf{E}^{(1)}) \cong B$.

Before carrying on the 'step-by-step' construction of the free simplicial algebra , we will interpret the first homotopy module $\pi_1(\mathbf{E}^{(1)})$ of $\mathbf{E}^{(1)}$ to find what it looks like.

For any simplicial algebra $\mathbf{E}$, if $\mathbf{E} = \mathbf{E}^{(1)}$, then

$$\pi_1(\mathbf{E}) = \mathrm{Ker}(\mathrm{Ker}d_0^1/\mathrm{Ker}d_0^1\mathrm{Ker} \overset{d_1^1}{\longrightarrow} E_0).$$

Indeed, by definition, the first homotopy module looks like

$$\pi_1(\mathbf{E}) = (\mathrm{Ker}d_0^1 \cap \mathrm{Ker}d_1^1)/d_2^2(\mathrm{Ker}d_0^2 \cap \mathrm{Ker}d_1^2).$$

From chapter 2 in section 4, the denominator of this homotopy module is exactly

$$\partial_2(NE_2) = d_2^2(\mathrm{Ker}d_0^2 \cap \mathrm{Ker}d_1^2) = \mathrm{Ker}d_0^1\mathrm{Ker}d_1^1.$$

Consider the morphism

$$\delta : \mathrm{Ker}d_0^1/\partial_2(NE_2) \longrightarrow E_0,$$

where $\delta = d_1$ (restricted to $NE_1/\partial_2 NE_2$). This is a crossed module. NE_0 acts on $NE_1/\partial_2 NE_2$ by multiplication via s, i.e.,

$$\begin{array}{ccc} NE_1/\partial_2 NE_2 \times NE_0 & \longrightarrow & NE_1/\partial_2 NE_2 \\ (\overline{x}, y) & \longmapsto & \overline{x} \cdot y = \overline{s_0(y)x}, \end{array}$$

where $\overline{x}$ denotes the corresponding element of $NE_1/\partial_2 NE_2$ whilst $x \in NE_1$. Since $x(s_0 d_1 y) - xy = x(s_0 d_1 y - y) \in \mathrm{Ker}d_0\mathrm{Ker}d_1 = \partial_2 NE_2$ with $x,\ y \in NE_1$, one can readily see that δ is the crossed module. Indeed, we show that the Peiffer condition for crossed module is satisfied, as follows:

For all $x + \partial_2 NE_2, y + \partial_2 NE_2$ with $x, y \in NE_1$,

$$\begin{array}{rcll} \delta(x + \partial_2 NE_2) \cdot (y + \partial_2 NE_2) & = & \delta(x) \cdot (y + \partial_2 NE_2), & \\ & = & d_1(x) \cdot y + \partial_2 NE_2, & \\ & = & s_0 d_1(x) y + \partial_2 NE_2 & \text{by the action,} \\ & \equiv & xy + \partial_2 NE_2 & \mod \partial_2 NE_2, \\ & = & (x + \partial_2 NE_2)(y + \partial_2 NE_2), & \end{array}$$

as required.

Finally, using Ker(d_1 :Ker$d_0 \to E_0$) =Ker$d_0 \cap$Kerd_1, one obtains

$$\begin{aligned}\pi_1(\mathbf{E}) &= \mathrm{Ker}(\mathrm{Ker}d_0^1/\partial_2(NE_2) \longrightarrow E_0) \\ &= \mathrm{Ker}(NE_1/\mathrm{Ker}d_0\mathrm{Ker}d_1 \longrightarrow E_0).\end{aligned}$$

In general, we may say that if $\mathbf{E}^{(k)}$ is the k-skeleton of the free simplicial algebra, then for $k \geq 1$,

$$\pi_k(\mathbf{E}^{(k)}) = \mathrm{Ker}(NE_k^{(k)}/\partial_{k+1}(NE_{k+1}^{(k+1)}) \longrightarrow E_{k-1}).$$

Proposition 3.4.2 *For any* $\mathbf{E}$ *with* $\mathbf{E} = \mathbf{E}^{(1)}$, $\partial_2(NE_2)$ *is generated by the Peiffer elements.*

Proof: By the case $n = 2$ in chapter 2, we have $\partial_2(NE_2) = \mathrm{Ker}d_0^1\mathrm{Ker}d_1^1$ and from lemma 3.4.1, we have

$$\begin{aligned}\mathrm{Ker}d_1^1 &= (X_1 - x_1, \ldots, X_n - x_n), \\ \mathrm{Ker}d_0^1 &= (X_1, \ldots, X_n).\end{aligned}$$

Thus Kerd_0Kerd_1 is an ideal generated by the elements of the form

$$(X_i - x_i)X_j \quad \text{with } 1 \leq i, j \leq n$$

which are the Peiffer elements. In other words, take generator elements $(X_i - s_0d_1X_i)X_j$ of $\partial_2(NE_2)$ and then

$$\begin{aligned}(X_i - s_0d_1X_i)X_j &= (X_i - s_0x_i)X_j \\ &= (X_i - x_i)X_j\end{aligned}$$

as $d_1(X_i) = x_i$. □

Proposition 3.4.3 *Given a presentation* $P = (R\,; x_1, \ldots, x_n)$ *of an R-algebra* B *and* $\mathbf{E}^{(1)}$ *the 1-skeleton of the free simplicial algebra generated by this presentation, then*

$$\delta : NE_1^{(1)}/\partial_2(NE_2^{(1)}) \longrightarrow NE_0^{(1)}$$

is the free crossed module on $\{x_1, \ldots, x_n\} \to R$*. In particular,*

$$\pi_1(\mathbf{E}^{(1)}) \cong \mathrm{Ker}(C \longrightarrow R)$$

where $C \cong R^n/\mathrm{Im}d$.

Proof: As we noted earlier, there is an equality

$$\pi_1(\mathbf{E}^{(1)}) = \mathrm{Ker}(NE_1/\mathrm{Ker}d_0\mathrm{Ker}d_1 \to E_0).$$

It follows from lemma 3.4.1 that

$$NE_1^{(1)} = \mathrm{Ker}d_0^1 = R^+[X_1, \ldots, X_n].$$

Moreover by the previous proposition, $\partial_2(NE_2^{(1)})$ =Kerd_0Kerd_1 is generated by the Peiffer elements of the form

$$(X_i - x_i)X_j \quad \text{with } 1 \leq i, j \leq n.$$

From theorem 1.4.2, we can thus define a free crossed module

$$\delta : R^+[X_1, \ldots, X_n]/\mathrm{Ker}d_0^1\mathrm{Ker}d_1^1 \longrightarrow R.$$

Any polynomial in $R^+[X_1, \ldots, X_n]$ is congruent modulo Kerd_0^1Kerd_1^1 to a monomial, i.e., an element in $R^{\{X_1,\ldots,X_n\}}$, the free module R^n with basis $X_1, \ldots, X_n$. This module has an algebra structure up to equivalence

$$X_iX_j \equiv x_iX_j \equiv x_jX_i \mod P_1.$$

Putting

$$C \cong R^+[X_1, \ldots, X_n]/\mathrm{Ker}d_0^1\mathrm{Ker}d_1^1.$$

and applying proposition 1.5.2 which gives

$$C = R^n/\mathrm{Im}d.$$

where $d : \Lambda^2 R^n \to R^n$ is the usual Koszul differential carrying the element $X_i \wedge X_j$ to $\varphi(X_i)X_j - X_i\varphi(X_j)$, $\varphi : R^n \to R$ given by $\varphi(X_i) = x_i$. □

Thus 3.4.3 gives the following corollary:

Corollary 3.4.4

$$\pi_1(\mathbf{E}^{(1)}) = \mathrm{Ker}(R^n/\mathrm{Im}d \longrightarrow R).$$

We now will recall the next step of the construction of a free simplicial algebra. Firstly we took a set of generators

$$\Omega^1 = \{y_1, \ldots, y_m\} \subset \pi_1(\mathbf{E}^{(1)})$$

and kill off the elements in the homotopy module $\pi_1(\mathbf{E}^{(1)})$ by adding new indeterminates $Y = \{Y_1, \ldots, Y_m\}$ into $E_2^{(1)}$ to establish

$$E_2^{(2)} = E_2^{(1)}[Y] = (R[s_0(X), s_1(X)])[Y].$$

together with

$$d_0^2(Y_i) = 0, \quad d_1^2(Y_i) = 0, \quad d_2^2(Y_i) = y_i.$$

Hence the 2-skeleton $\mathbf{E}^{(2)}$ looks like

$$\mathbf{E}^{(2)} : \ldots (R[s_0(X), s_1(X)])[Y] \underset{s_0, s_1}{\overset{d_0, d_1, d_2}{\rightrightarrows}} R[X] \underset{s_0}{\overset{d_0, d_1}{\rightrightarrows}} R \xrightarrow{\ f\ } R/I.$$

For $\mathbf{E}^{(2)}$, higher levels than dimension 2 are generated by degeneracy elements.

3.5 Free Crossed Resolutions

The reason for giving the previous section is the following construction.

A 'step-by-step' construction of a free simplicial algebra is constructed from simplicial algebra inclusions

$$\mathbf{E}^{(0)} \subseteq \mathbf{E}^{(1)} \subseteq \mathbf{E}^{(2)} \subseteq \ldots$$

In the following, we take the functor $\mathbf{C}$, which is described in section 3.3, to see what $C_k(\mathbf{E}^{(k)})$ looks like, where $\mathbf{E}^{(k)}$ is the k-skeleton of that construction.

Recall the 'step-by-step' construction of the free simplicial algebra $\mathbf{E}$

For $k = 0$, there is the 0-skeleton $\mathbf{E}^{(0)}$ of the construction

$$\ldots \quad R \longrightarrow R \longrightarrow R/(x_1, \ldots, x_n).$$

Here $\mathbf{E}^{(0)}$ is the trivial simplicial algebra in which in every degree n, $E_n^{(0)} = R$ and $d_i^n = \mathrm{id} = s_j^n$.

It is easy to see that $C_0(\mathbf{E}^{(0)}) = R$ as $NE_1 \cap D_1$ is trivial.

The 1-skeleton, for $k = 1$, $\mathbf{E}^{(1)}$ is

$$\mathbf{E}^{(1)} : \ldots R[s_0(X), s_1(X)] \underset{s_0, s_1}{\overset{d_0, d_1, d_2}{\rightrightarrows}} R[X] \underset{s_0}{\overset{d_0, d_1}{\rightrightarrows}} R \xrightarrow{\ f\ } R/I.$$

and since $E_2^{(1)}$ is generated by the degeneracy elements, $E_2^{(1)} = D_2$. So the crossed complex term $C_1(\mathbf{E}^{(1)})$ is the following

$$\begin{aligned} C_1(\mathbf{E}^{(1)}) &= \frac{NE_1^{(1)}}{[NE_1^{(1)} \cap D_1 + \partial_2(NE_2^{(1)} \cap D_2)],} \\ &= \frac{NE_1^{(1)}}{\partial_2(NE_2^{(1)} \cap D_2)} \qquad \text{since } NE_1 \cap D_1 = 0, \\ &= \frac{NE_1^{(1)}}{\partial_2(NE_2^{(1)})} \qquad \text{as } E_2^{(1)} = D_2. \end{aligned}$$

By lemma 3.4.1 and proposition 3.4.2, we have $NE_1^{(1)} = R^+[X_1, \ldots, X_n]$ and $\partial_2(NE_2^{(1)})$ is generated by the Peiffer elements, respectively. It then follows that

$$C_1(\mathbf{E}^{(1)}) = R^+[X_1, \ldots, X_n]/P_1.$$

Here P_1 is the first order Peiffer ideal. The proof of theorem 1.4.2 shows that

$$\partial_1 : R^+[X_1, \ldots, X_n]/P_1 \longrightarrow R$$

is a free crossed module.

Looking at the case 2, the 2-skeleton of the construction is

$$\mathbf{E}^{(2)} : \ldots (R[s_0(X), s_1(X)])[Y] \underset{s_0, s_1}{\overset{d_0, d_1, d_2}{\rightrightarrows}} R[X] \underset{s_0}{\overset{d_0, d_1}{\rightrightarrows}} R \xrightarrow{\ f\ } R/I.$$

As before $E_3^{(2)} = D_3$ as $E_3^{(2)}$ is generated by the degeneracy elements. Thus the second term of crossed complex is

$$\begin{aligned} C_2(\mathbf{E}^{(2)}) &= \frac{NE_2^{(2)}}{[NE_2^{(2)} \cap D_2 + \partial_3(NE_3^{(2)} \cap D_3)],} \\ &= \frac{NE_2^{(2)}}{[NE_2^{(2)} \cap D_2 + \partial_3(NE_3^{(2)})]} \qquad \text{as } E_3^{(2)} = D_3. \end{aligned}$$

If $x, y \in NE_1$, then $NE_2 \cap D_2$ is generated by the elements of the form

$$s_1 x(s_0 y - s_1 y)$$

and in general, if $x, y \in NE_{n-1}$, then

$$s_{n-1} x(s_{n-2} y - s_{n-1} y) \in NE_n \cap D_n.$$

For the case of $\mathbf{E}^{(2)}$, if X_i and X_j are in $NE_1^{(2)}$, then the generators of the ideal $NE_2^{(2)} \cap D_2$ are of the form

$$s_1X_i(s_0X_j - s_1X_j).$$

Now look at $\partial_3(NE_3^{(2)})$ in terms of the skeleton $\mathbf{E}^{(2)}$. In a similar way to the proof of lemma 3.4.1 and to $d_0^2(Y_i) = d_1^2(Y_i) = 0$, one can readily obtain the following:

$$NE_2^{(2)} = (R[s_0(X), s_1(X)])^+[Y].$$

On the other hand, Chapter 2 provides the following result which is

$$\partial_3(NE_3^{(2)}) = \sum_{\{I,J\}} K_I K_J + K_{\{0,1\}}K_{\{0,2\}} + K_{\{0,2\}}K_{\{1,2\}} + K_{\{0,1\}}K_{\{1,2\}}$$

where $I \cup J = [2],\ I \cap J = \emptyset$ and

$$\begin{aligned}
K_{\{0,1\}}K_{\{0,2\}} &= (\mathrm{Ker}d_0 \cap \mathrm{Ker}d_1)(\mathrm{Ker}d_0 \cap \mathrm{Ker}d_2)\\
K_{\{0,2\}}K_{\{1,2\}} &= (\mathrm{Ker}d_0 \cap \mathrm{Ker}d_2)(\mathrm{Ker}d_1 \cap \mathrm{Ker}d_2)\\
K_{\{0,1\}}K_{\{1,2\}} &= (\mathrm{Ker}d_0 \cap \mathrm{Ker}d_1)(\mathrm{Ker}d_1 \cap \mathrm{Ker}d_2)
\end{aligned}$$

which are generated by the following elements, for $X_i \in NE_1 = \mathrm{Ker}d_0$ and $Y_i \in NE_2 = \mathrm{Ker}d_0 \cap \mathrm{Ker}d_1$

$$\begin{gathered}
(s_1s_0d_1X_i - s_0X_i)Y_j,\\
(s_0X_i - s_1X_i)(s_1d_2Y_j - Y_j),\\
s_1X_i(s_0d_2Y_j - s_1d_2Y_j + Y_j);
\end{gathered}$$

and for Y_i and $Y_j \in NE_2$ with $1 \leq i, j \leq n$

$$\begin{gathered}
Y_i(s_1d_2Y_j - Y_j),\\
Y_i(Y_j + s_0d_2Y_j - s_1d_2Y_j),\\
(s_0d_2Y_i - s_1d_2Y_i + Y_i)(s_1d_2Y_j - Y_j).
\end{gathered}$$

Rewrite these elements as follows:

$$(s_1s_0d_1X_i - s_0X_i)Y_j \qquad (i)$$
$$Y_i(s_1d_2Y_j - Y_j) \qquad (ii)$$
$$(s_0X_i - s_1X_i)(s_1d_2Y_j - Y_j) \qquad (iii)$$
$$Y_i(Y_j + s_0d_2Y_j - s_1d_2Y_j) \qquad (iv)$$
$$s_1X_i(s_0d_2Y_j - s_1d_2Y_j + Y_j) \qquad (v)$$
$$(s_0d_2Y_i - s_1d_2Y_i + Y_i)(s_1d_2Y_j - Y_j) \qquad (vi).$$

The ideal generated by these elements will be denoted by P_2 and will be called the second order Peiffer ideal. In the next chapter, we will explicitly interpret these second order Peiffer elements.

Thus we can immediately state the subsequent proposition:

Proposition 3.5.1 *For any simplicial algebra* $\mathbf{E}$, *if* $\mathbf{E} = \mathbf{E}^{(2)}$, *then the image of the third term of the Moore complex of* $\mathbf{E}^{(2)}$ *is generated by the second order Peiffer elements* P_2.

Finally writing $Q_2 = NE_2^{(2)} \cap D_2$, we get the second term of crossed complex as follows

$$C_2(\mathbf{E}^{(2)}) = \frac{(R[s_0(X), s_1(X)])^+[Y]}{Q_2 + P_2.}$$

We thus can form:

Proposition 3.5.2 *Let* $\mathbf{E}^{(2)}$ *be the 2-skeleton of a free simplicial algebra. Then*

$$\mathscr{C}^{(2)}: \quad (R[s_0(X), s_1(X)])^+[Y]/[Q_2 + P_2] \xrightarrow{\partial_2} R^+[X]/P_1 \xrightarrow{\partial_1} R \xrightarrow{f} R/I \xrightarrow{g} 0$$

is the k-skeleton of a free crossed resolution of $R/(x_1, \ldots, x_n)$, *where* ∂_2 *and* ∂_1 *are given by respectively, for* $Y_i \in (R[s_0(X), s_1(X)])^+[Y]$ *and* $X_i \in R[X]^+$,

$$\partial_2[Y_i + (Q_2 + P_2)] = \partial_2(Y_i) + P_1 \text{ and } \partial_1(X_i + P_1) = \partial_1(X_i).$$

Proof: This follows immediately from the particular case of the step-by-step construction of the free simplicial algebra. □

Conjecture: If $\mathbf{E}^{(k)}$ is the k-skeleton of the construction of the free simplicial resolution, then

$$\mathscr{C}^{(k)}: \quad NE_k^{(k)}/[Q_k + P_k] \xrightarrow{\partial_k} \cdots \xrightarrow{\partial_2} NE_1^{(1)}/P_1 \xrightarrow{\partial_1} R \xrightarrow{f} R/I \to 0$$

is the k-skeleton of a free crossed resolution of R/I, where P_k is the kth order of Peiffer ideal in $NE_k^{(k)}$ and $Q_k = NE_k^{(k)} \cap D_k$.

CHAPTER 4

2-CROSSED MODULES AND THE N-TYPE OF THE K-SKELETON

4.1 2-CROSSED MODULES OF ALGEBRAS

As was mentioned in chapter 2, crossed modules were initially defined by Whitehead as models for 2-types. D.Conduché, [14], in 1984 described the notion of 2-crossed module as a model for 3-types

In this section, we describe a 2-crossed module and a free 2-crossed module of algebras by using the second order Peiffer elements. The following definition of 2-crossed modules of commutative algebras was given by A.R.Grandjeán and M.J.Vale [24].

Definition 4.1.1 *A 2-crossed module of* $\mathbf{k}$*-algebras consists of a complex of* C_0*-algebras*

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

and ∂_2, ∂_1 *morphisms of* C_0*-algebras, where the algebra* C_0 *acts on itself by multiplication such that*

$$C_2 \xrightarrow{\partial_2} C_1$$

is a crossed module. Thus C_1 *acts on* C_2 *via* C_0 *and we require that for all* $x \in C_2,\ y \in C_1$ *and* $z \in C_0$ *that* $(xy)z = x(yz)$*. Further, there is a* C_0*-bilinear function giving*

$$\{\quad \otimes \quad\} : C_1 \otimes_{C_0} C_1 \longrightarrow C_2,$$

called a Peiffer lifting, which satisfies the following axioms:

$$\begin{array}{lrcl}
PL1: & \partial_2\{y_0 \otimes y_1\} & = & y_0 y_1 - y_0 \cdot \partial_1(y_1), \\
PL2: & \{\partial_2(x_1) \otimes \partial_2(x_2)\} & = & x_1 x_2, \\
PL3 & \{y_0 \otimes y_1 y_2\} & = & \{y_0 y_1 \otimes y_2\} + \partial_1 y_2 \cdot \{y_0 \otimes y_1\} \\
PL4: \ a) & \{\partial_2(x) \otimes y\} & = & y \cdot x - \partial_1(y) \cdot x, \\
\qquad b) & \{y \otimes \partial_2(x)\} & = & y \cdot x, \\
PL5: & \{y_0 \otimes y_1\} \cdot z & = & \{y_0 \cdot z \otimes y_1\} = \{y_0 \otimes y_1 \cdot z\},
\end{array}$$

for all $x, x_1, x_2 \in C_2,\ y, y_0, y_1, y_2 \in C_1$ *and* $z \in C_0$.

We denote such a 2-crossed module of algebras by $\{C_2,\, C_1,\, C_0,\, \partial_2,\, \partial_1\}$.

Note that since $\{\quad \otimes \quad\}$ is C_0-bilinear, we have the equalities:

$$\begin{array}{rcl}
\{y_0 \otimes (y_1 + y_2)\} & = & \{y_0 \otimes y_1\} + \{y_0 \otimes y_2\}, \\
\{(y_0 + y_1) \otimes y_2\} & = & \{y_0 \otimes y_2\} + \{y_1 \otimes y_2\}.
\end{array}$$

A morphism of 2-crossed modules of algebras may be pictured by the diagram

$$\begin{array}{ccccc}
C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 \\
\downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\
C_2^{'} & \xrightarrow{\partial_2^{'}} & C_1^{'} & \xrightarrow{\partial_1^{'}} & C_0^{'}
\end{array}$$

such that $f_0 \partial_1 = \partial_1' f_1,\ f_1 \partial_2 = \partial_2' f_2$ and such that

$$f_1(c_0 \cdot c_1) = f_0(c_0) \cdot f_1(c_1), \quad f_2(c_0 \cdot c_2) = f_0(c_0) \cdot f_2(c_2),$$

and

$$\{\quad \otimes \quad\} f_1 \otimes f_1 = f_2\{\quad \otimes \quad\},$$

for all $c_2 \in C_2,\ c_1 \in C_1,\ c_0 \in C_0$.

We thus define the category of 2-crossed module denoting it as $\mathbf{X_2Mod}$.

Morphisms f_1 and f_2 are called *equivariant* if $C_0 = C_0'$ with $f_0 =$identity of C_0.

The following theorems, in some sense, are well known in algebraic setting such as group, Lie algebras. Thus we do not give all details of the proofs as analogous proofs can be found in the literature [22], [14] and the adaptation to the case of commutative algebras is routine

We show that the usefulness of $\partial_n NE_n$ of order 2 gives the following theorem:

Theorem 4.1.2 *The category of crossed modules is equivalent to the category of simplicial algebras with Moore complex of length* 1.

Proof: Let **E** be a simplicial algebra with Moore complex of length 1. Put

$$M = NE_1, \; N = NE_0 \text{ and } \partial_1 = d_1 \text{ (restricted to } M).$$

Then NE_0 acts on NE_1 by multiplication via s_0. Since the Moore complex is of length 1, we have

$$\partial_2 NE_2 = \text{Ker}d_0 \text{Ker}d_1 = 0$$

and the generators of this ideal are of the form $x(s_0d_1y - y)$ with $x, y \in NE_1$ (see section 2.4.1). It then follows that for all $x, x' \in M$,

$$\begin{aligned} \partial_1(x) \cdot x' &= d_1(x) \cdot x' \\ &= s_0 d_1(x) x' && \text{by the action,} \\ &= x x' && \text{since } \partial_2 NE_2 = 0. \end{aligned}$$

Thus $\partial_1 : M \to N$ is a crossed module. This yields a functor

$$\mathbf{N_1} : \mathbf{SimpAlg} \longrightarrow \mathbf{XMod}$$

Conversely, let $\partial_1 : M \to N$ be a crossed module. By using the action of N on M, one forms the semidirect product $M \rtimes N$ together with homomorphisms

$$d_0(m,n) = n, \; d_1(m,n) = \partial_1 m + n, \; s_0(n) = (0,n).$$

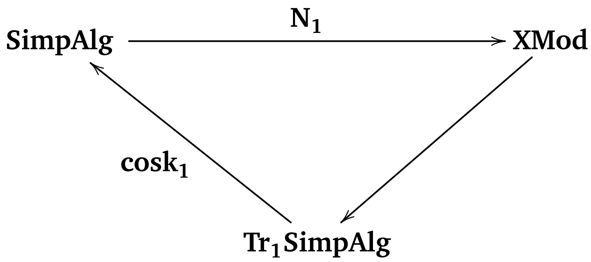
Define

$$E_0 = N, \qquad E_1 = M \rtimes N.$$

Then, we have a 1-truncated simplicial algebra

$$\{E_0, E_1\}.$$

There is a $\mathbf{cosk}_1$ functor from the category of 1-truncated simplicial algebras to that of simplicial algebras. Thus we have the following diagram



and this enables us to define a functor

$$\mathbf{S}_1 : \mathbf{XMod} \longrightarrow \mathbf{SimpAlg}$$

Using lemma 1.1.6, **E** is a simplicial algebra whose Moore complex is of length 1. The correspondence gives rise to an equivalence of categories. □

The reason for giving this theorem is to generalise it. Before that we present some results.

Let **E** be a simplicial algebra with Moore complex **NE** and let **NE**′ be the truncation of the Moore complex **NE** of order 2

$$\mathbf{NE}' : \qquad NE_2 \xrightarrow{\partial_2} NE_1 \xrightarrow{\partial_1} NE_0.$$

Writing

$$\begin{aligned} L &= NE_2 = \mathrm{Ker}d_0 \cap \mathrm{Ker}d_1, \\ M &= NE_1 = \mathrm{Ker}d_0, \\ N &= NE_0 = E_0, \end{aligned}$$

with $NE_3' = 0$. Using the generator elements of $\partial_3(NE_3') = 0$, one gets the following

$$\begin{aligned} l(s_1s_0d_1m - s_0m) &\in (\mathrm{Ker}d_0 \cap \mathrm{Ker}d_1)\mathrm{Ker}d_2, \\ l_1(s_1d_2l_0 - l_0) &\in (\mathrm{Ker}d_0 \cap \mathrm{Ker}d_1)(\mathrm{Ker}d_0 \cap \mathrm{Ker}d_2), \\ (s_1d_2l - l)(s_0m - s_1m) &\in (\mathrm{Ker}d_0 \cap \mathrm{Ker}d_2)\mathrm{Ker}d_1, \\ (l + s_0d_2l - s_1d_2l)s_1m &\in (\mathrm{Ker}d_1 \cap \mathrm{Ker}d_2)\mathrm{Ker}d_{0.} \end{aligned}$$

and these imply the equalities:

$$\begin{aligned} l(s_1s_0d_1m - s_0m) &= 0 \qquad (1)\\ l_1(s_1d_2l_0 - l_0) &= 0 \qquad (2)\\ (s_0m - s_1m)(s_1d_2l - l) &= 0 \qquad (3)\\ s_1m(l + s_0d_2l - s_1d_2l) &= 0 \qquad (4) \end{aligned}$$

with $l_1, l_0, l \in L$ and $m \in M$.

Consider the following diagram of morphisms

$$\begin{array}{ccccc} & & M \otimes M & & \\ & \swarrow & & \searrow^{\rho} & \\ L & & \xrightarrow{\partial_2} & & M \xrightarrow{\partial_1} N \end{array}$$

The algebra M acts, in two ways, on the algebra L: by multiplication via s_0 and via s_1 in E_2. The action via s_0 will be denoted by $l \cdot m = s_0(m)l$ and the action via s_1 will be denoted by $m \cdot l = s_1(m)l$. The action of N on L is given as follows:

from equality (1), there is a commutative diagram

$$\begin{array}{ccc} L \otimes M & \xrightarrow{\cdot} & L \\ {\scriptstyle id \otimes \partial_1} \downarrow & & \downarrow {\scriptstyle id} \\ L \otimes N & \longrightarrow & L \end{array}$$

given by

$$\begin{array}{ccc} (l \otimes m) & \longmapsto & l \cdot m = s_0(m)l \\ \downarrow & & \downarrow \\ (l \otimes \partial_1 m) & \longmapsto & l \cdot \partial_1(m) = (s_1s_0d_1m)l, \end{array}$$

which gives an equality

$$\partial_1(m) \cdot l = s_1s_0\partial(m)l = s_0(m)l = l \cdot m \qquad (*).$$

Define the map ρ, for $m_0, m_1 \in M$,

$$\rho(m_0 \otimes m_1) = m_0 m_1 - m_0 \cdot \partial_1 m_1$$

that is the Peiffer element in M corresponding to $\{m_0 \otimes m_1\}$. Thus $\partial_1 : M \to N$ is a crossed module if this map ρ is zero. Next identify the map $M \otimes M \to L$ by using P.Carrasco's idea to interpret the Peiffer lifting map

$$\{ \quad \otimes \quad \} : M \otimes M \longrightarrow L.$$

This is correspond to the map which is defined in section 2.3.,

$$-C_{(1)(0)} \ : NE_1 \otimes NE_1 \longrightarrow NE_2$$

given by

$$C_{(1)(0)} \ (m_0 \otimes m_1) = p(s_1(m_0)s_0(m_1)) = p_1 p_0(s_1(m_0)s_0(m_1))$$

and we thus readily obtain

$$-C_{(1)(0)}(m_0 \otimes m_1) = s_1 m_0 (s_1 m_1 - s_0 m_1).$$

We will show that the generating elements of $\partial_3(NE_3 \cap D_3)$ pays off the following result.

Proposition 4.1.3 *Let* **E** *be a simplicial algebra with the Moore complex* **NE.** *Then the complex of algebras*

$$NE_2/\partial_3(NE_3 \cap D_3) \xrightarrow{\overline{\partial}_2} NE_1 \xrightarrow{\partial_1} NE_0$$

is a 2-crossed module of algebras, where the Peiffer map is defined as follows:

$$\begin{array}{rccc} \{ \quad \otimes \quad \} : & NE_1 \otimes NE_1 & \longrightarrow & NE_2/\partial_3(NE_3 \cap D_3) \\ & (y_0 \otimes y_1) & \longmapsto & \overline{s_1 y_0 (s_1 y_1 - s_0 y_1)}. \end{array}$$

Here the right hand side denotes a coset in $NE_2/\partial_3(NE_3 \cap D_3)$ represented by an element in NE_2.

Proof: We will show that all axioms of a 2-crossed module are verified. It is readily checked that the morphism $\overline{\partial}_2 : NE_2/\partial_3(NE_3 \cap D_3) \to NE_1$ is a crossed module (see proposition 5.1.4). In the following calculations we display the elements omitting the overlines as:

PL1:

$$\begin{aligned}\overline{\partial}_2\{y_0 \otimes y_1\} &= \partial_2(s_1 y_0(s_1 y_1 - s_0 y_1)) \\ &= d_2 s_1 y_0(d_2 s_1 y_1 - d_2 s_0 y_1) \\ &= y_0(y_1 - s_0 d_1 y_1) \\ &= y_0 y_1 - y_0(s_0 d_1) y_1 \\ &= y_0 y_1 - y_0 \cdot \partial_1 y_1.\end{aligned}$$

PL2: From $\partial_3(C_{(1)(0)}(x_1 \otimes x_2)) = s_1 d_2(x_1) s_0 d_2(x_2) - s_1 d_2(x_1) s_1 d_2(x_2) + x_1 x_2$ (see p. 46), one obtains

$$\begin{aligned}\{\overline{\partial}_2(x_1) \otimes \overline{\partial}_2(x_2)\} &= s_1 d_2 x_1(s_1 d_2 x_2 - s_0 d_2 x_2) \\ &\equiv x_1 x_2 \mod \partial_3(NE_3 \cap D_3).\end{aligned}$$

PL3:

$$\begin{aligned}\{y_0 \otimes y_1 y_2\} &= s_1 y_0[s_1(y_1 y_2) - s_0(y_1 y_2)] \\ &= s_1 y_0[s_1(y_1) s_1(y_2) - s_1(y_1) s_0(y_2) + s_1(y_1) s_0(y_2) - s_0(y_1) s_0(y_2)] \\ &= s_1 y_0[s_1 y_1(s_1 y_2 - s_0 y_2)] + [s_1 y_0(s_1 y_1 - s_0 y_1)] s_0 y_2 \\ &= s_1(y_0 y_1)(s_1 y_2 - s_0 y_2) + \{y_0 \otimes y_1\} s_0 y_2\end{aligned}$$

but $\partial_3(C_{(1,0)(2)}(y \otimes x)) = (s_1 s_0 d_1 y - s_0 y)x$, so this implies

$$\begin{aligned}\{y_0 \otimes y_1 y_2\} &\equiv s_1(y_0 y_1)(s_1 y_2 - s_0 y_2) + s_1 s_0 d_1(y_2)\{y_0 \otimes y_1\} \mod \partial_3(NE_3 \cap D_3) \\ &= \{y_0 y_1 \otimes y_2\} + \partial_1 y_2 \cdot \{y_0 \otimes y_1\} \text{ by the definition of the action.}\end{aligned}$$

PL4: a)

$$\{\overline{\partial}_2(x) \otimes y\} = s_1 \partial_2 x(s_1 y - s_0 y),$$

but

$$\partial_3(C_{(2,0)(1)}(y \otimes x)) = (s_0 y - s_1 y) s_1 d_2 x - (s_0 y - s_1 y)x \in \partial_3(NE_3 \cap D_3)$$

and

$$\partial_3(C_{(1,0)(2)}(y \otimes x)) = (s_1 s_0 d_1 y - s_0 y)x \in \partial_3(NE_3 \cap D_3),$$

(see p. 45 and 44) so then

$$\begin{aligned}\{\overline{\partial}_2(x) \otimes y\} &\equiv s_1(y)x - s_0(y)x && \mod \partial_3(NE_3 \cap D_3) \\ &\equiv s_1(y)x - s_1 s_0 d_1(y)x && \mod \partial_3(NE_3 \cap D_3) \\ &= y \cdot x - \partial_1(y) \cdot x && \text{by the definition of the action,}\end{aligned}$$

b) since $\partial_3(C_{(2,1)(0)}(y \otimes x)) = s_1 y(s_0 d_2 x - s_1 d_2 x) + s_1(y)x$,

$$\begin{aligned}\{y \otimes \overline{\partial}_2(x)\} &= s_1 y(s_1 \partial_2 x - s_0 \partial_2 x) \\ &\equiv s_1(y)x \quad \text{mod } \partial_3(NE_3 \cap D_3) \\ &= y \cdot x \qquad \text{by the definition of the action.}\end{aligned}$$

PL5:

$$\begin{aligned}\{y_0 \otimes y_1\} \cdot z &= (s_1 y_0(s_1 y_1 - s_0 y_1)) \cdot z \\ &= s_1 s_0(z) s_1(y_0)(s_1 y_1 - s_0 y_1) \\ &= s_1(s_0(z) y_0)(s_1 y_1 - s_0 y_1) \\ &= s_1(y_0 \cdot z)(s_1 y_1 - s_0 y_1) \quad \text{by the definition of the action} \\ &= \{y_0 \cdot z \otimes y_1\}.\end{aligned}$$

Clearly the same sort of argument works for

$$\{y_0 \cdot z \otimes y_1\} = \{y_0 \otimes y_1 \cdot z\}$$

with $x, x_1, x_2 \in NE_2/\partial_3(NE_3 \cap D_3)$, $y, y_0, y_1, y_2 \in NE_1$ and $z \in NE_0$. This completes the proof of the proposition. □

This proposition gives the generalisation of theorem 4.1.2 as follows. The methods we use for proving the subsequent result are based on ideas of Ellis, [22]. A different prove of this result for the commutative algebraic version is noted in [24].

Theorem 4.1.4 *The category of 2-crossed modules is equivalent to the category of simplicial algebras with Moore complex of length* 2*.*

Proof: Let **E** be a simplicial algebra with Moore complex of length 2. In the previous proposition, a 2-crossed module

$$NE_2 \xrightarrow{\partial_2} NE_1 \xrightarrow{\partial_1} NE_0$$

has already been constructed. Thus there exists an obvious functor

$$\mathbf{N_2} : \mathbf{SimpAlg} \longrightarrow \mathbf{X_2Mod}$$

Conversely suppose given a 2-crossed module

$$L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N.$$

Define $E_0 = N$. We can create the semidirect product $E_1 = M \rtimes N$ by using the action of N on M together with homomorphisms

$$d_0(m,n) = n,\ d_1(m,n) = \partial_1 m + n,\ s_0(n) = (0,n),$$

By using the axioms $a)$ and $b)$ of the $PL3$, there is an action of $m \in M$ on $l_1 \in L$ given by

$$m \cdot l_1 = \partial_1 m \cdot l_1 - \{\partial_2 l_1 \otimes m\}.$$

Using this action we form the semidirect product $L \rtimes M$. An action of $(m,n) \in M \rtimes N$ on $(l_1, m_1) \in L \rtimes M$ is given by

$$(m,n) \cdot (l_1, m_1) = (m \cdot l_1 + n \cdot l_1, mm_1 + n \cdot m).$$

Using this action we get the semidirect product

$$E_2 = (L \rtimes M) \rtimes (M \rtimes N).$$

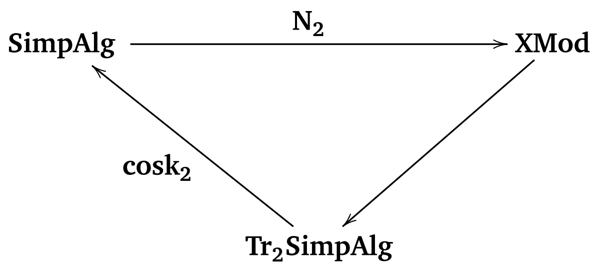
(The bilinearity of $\{\ \ \otimes\ \ \}$ to together with axioms $PL3$ and $PL5$ ensure that these last two actions are indeed commutative actions.) There are homomorphisms

$$\begin{array}{rclrcl} d_0(l_1,m_1,m_2,n) &=& (m_2,n) & s_0(m_2,n) &=& (0,0,m_2,n), \\ d_1(l_1,m_1,m_2,n) &=& (m_1+m_2,n) & s_1(m_2,n) &=& (0,m_2,0,n). \\ d_2(l_1,m_1,m_2,n) &=& (m_1,\partial_1 m_2 + n), & & & \end{array}$$

We have a 2-truncated simplicial algebra

$$\{E_0, E_1, E_2\}.$$

There is a $\mathbf{cosk}_2$ functor from the category of 2-truncated simplicial algebras to that of simplicial algebras. Thus we have the following diagram



and this enables us to define a functor

$$\mathbf{S_2} : \mathbf{X_2Mod} \longrightarrow \mathbf{SimpAlg}$$

Using lemma 1.1.6, **E** is a simplicial algebra whose Moore complex is of length 2. This correspondence gives rise to an equivalence of categories completing the proof of the theorem. □

4.1.1 Free 2-Crossed Modules

The definition of a free 2-crossed module is similar in some ways to the corresponding definition of a free crossed module. However, the construction of a free 2-crossed module is a bit more complicated and is given by means of the 2-skeleton of a free simplicial algebra.

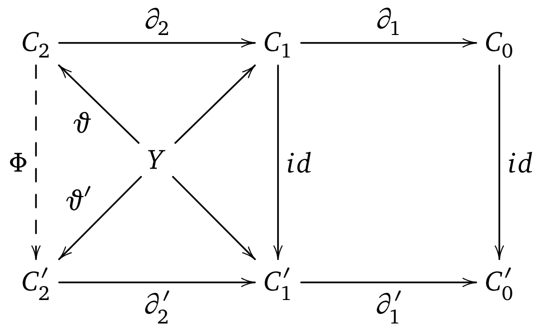
Definition 4.1.5 *Let $\{C_2, C_1, C_0, \partial_2, \partial_1\}$ be a 2-crossed module, let Y be a set and let $\vartheta : Y \to C_2$, then $\{C_2, C_1, C_0, \partial_2, \partial_1\}$ is said to be a free 2-crossed module with basis ϑ or, alternatively, on the function $\partial_2\vartheta : Y \to C_1$ if for any 2-crossed module $\{C_2', C_1, C_0, \partial_2', \partial_1\}$ and function $\vartheta' : Y \to C_2'$ such that $\partial_2\vartheta = \partial_2'\vartheta'$, there is a unique morphism*

$$\Phi : C_2 \longrightarrow C_2'$$

such that $\partial_2'\Phi = \partial_2$.

The 2-crossed module $\{C_2, C_1, C_0, \partial_2, \partial_1\}$ is *totally free* if $\partial_1 : C_1 \to C_0$ is a totally free pre-crossed module.

This situation may be pictured as



We shall give an explicit description of the construction of a free 2-crossed module. For this, we need to recall the 2-skeleton of the free simplicial algebra which is

$$\mathbf{E}^{(2)}: \ldots (R[s_0(X)s_1(X)])[Y] \underset{s_0, s_1}{\overset{d_0, d_1, d_2}{\rightrightarrows}} R[X] \underset{s_0}{\overset{d_0, d_1}{\rightrightarrows}} R,$$

with the simplicial structure defined as in section 3.4.

Again, we will assume that Y and X are sets of m indeterminates $Y_1, \ldots, Y_m$ and n indeterminates, $X_1, \ldots, X_n$, respectively and we will assume $m, n < \infty$. Define a morphism

$$\psi : (R[s_0(X), s_1(X)])^+[Y] \longrightarrow R^+[X],$$

where ψ is induced by d_2 and define

$$\varphi : R^+[X] \longrightarrow R,$$

here φ is induced by d_1. We denote this 2-dimensional construction (see section 3.4) data by $(Y, X; \psi, \varphi, R)$.

The construction of a free 2-crossed module is as follows:

Theorem 4.1.6 *A totally free 2-crossed module* $\{L, E, R, \psi', \varphi\}$ *exists on the 2-dimensional construction data* $(Y, X; \psi, \varphi, R)$.

Proof: Suppose given the 2-dimensional construction data described above and given a function f from a set Y to $E = R^+[X]$, the positively graded part of the polynomial algebra over an $\mathbf{k}$-algebra R in the n indeterminates X_i.

Take $D = (R[s_0(X), s_1(X)])^+[Y]$, the positively graded part of the polynomial algebra on Y so that E acts on D by multiplication via s_1. f induces a morphism ψ of E-algebras

$$\begin{array}{ccc} D & \xrightarrow{\ \psi\ } & E \\ & \nwarrow \quad \nearrow_{f} & \\ & Y & \end{array}$$

defined on generators by $\psi(y) = f(y)$.

Let $\{A, E, R, \delta, \eta\}$ be any 2-crossed module and let $\vartheta' : Y \to A$ with $\delta\vartheta' = f$. Recall the second order Peiffer ideal P_2 in D. It is easily checked that $\psi(P_2) = 0$ as all generator elements of P_2 are in $\mathrm{Ker}d_2$. By taking the factor module $L = D/P_2$, there exists a morphism $\psi' : L \to E$ such that the diagram,

$$\begin{array}{ccc} D & \xrightarrow{\quad\psi\quad} & E \\ & {\scriptstyle q}\searrow \quad \nearrow{\scriptstyle \psi'} & \\ & L & \end{array}$$

commutes, where q is the quotient morphism of algebras. Thus ψ' is a crossed module. Indeed, given the elements $y + P_2$, $y' + P_2 \in L$,

$$\begin{aligned} \psi'(y + P_2) \cdot (y' + P_2) &= \psi(y) \cdot y' + P_2 \\ &= s_1 d_2(y) y' + P_2 \\ &\equiv y y' + P_2 \mod P_2 \\ &= (y + P_2)(y' + P_2). \end{aligned}$$

Hence there exists a unique morphism $\Phi : L \to A$ given by $\Phi(y + P_2) = \vartheta'(y)$ such that $\delta\Phi = \psi'$. That is

$$\begin{array}{ccccccc} Y & \xrightarrow{\vartheta} & L & \xrightarrow{\psi'} & E & \xrightarrow{\varphi} & R \\ \Vert & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} \\ Y & \xrightarrow[\vartheta']{} & A & \xrightarrow[\delta]{} & E & \xrightarrow[\eta]{} & R \end{array}$$

Therefore $\{L, E, R, \psi', \varphi\}$, or the complex

$$(R[s_0(X), s_1(X)])^+[Y]/P_2 \xrightarrow{\psi'} R^+[X] \xrightarrow{\varphi} R$$

is the required free 2-crossed module on $(Y, X; \psi, \varphi, R)$. Here P_2 is the second order Peiffer ideal in $(R[s_0(X), s_1(X)])^+[Y]$. The Peiffer lifting map

$$\{\ \ \otimes\ \ \} : R^+[X] \otimes R^+[X] \longrightarrow (R[s_0(X), s_1(X)])^+[Y]/P_2$$

is induced by the map

$$\omega : R^{+}[X] \otimes R^{+}[X] \longrightarrow (R[s_0(X), s_1(X)])^{+}[Y]$$

given by

$$\omega(X_i \otimes X_j) = s_1X_i(s_1X_j - s_0X_j) \quad \text{with } X_i, X_j \in R^{+}[X].$$

Thus we can define the Peiffer lifting map by

$$\{X_i \otimes X_j\} = \overline{\omega(X_i \otimes X_j)} = \overline{s_1X_i(s_1X_j - s_0X_j)}.$$

In a similar way to proposition 4.1.5, the 2-crossed module axioms can be checked. □

Note: In the group case, a closely related structure to that of 2-crossed module is that of *a quadratic module,* defined by H.J.Baues [5]. Although it seems intuitively clear that the results above should extend to an algebra version of quadratic modules. I have not managed to check all the details and so have omitted a study of this idea from this thesis.

4.2 The n-Type of the k-Skeleton

Recall from [1] that a morphism $\mathbf{f} : \mathbf{E} \to \mathbf{F}$ of simplicial algebras will be called an *n-equivalence* if

$$\pi_i(\mathbf{f}) : \pi_i(\mathbf{E}) \longrightarrow \pi_i(\mathbf{F}),$$

is an isomorphism for all i, $0 \leq i \leq n$. Two simplicial algebras $\mathbf{E}$ and $\mathbf{F}$ are said to have the same *n-type* if there is a chain of n-equivalences linking them. From proposition 1.2.3, a simplicial algebra $\mathbf{F}$ is an *n-type* if

$$\pi_i(\mathbf{F}) = 0 \quad \text{for } i > n.$$

In this section we show how the k-skeleton of a free simplicial algebra occurs in describing algebraic models of n-types.

4.2.1 1-Types

Assume given the 0-step of the construction of a free simplicial algebra of an R-algebra $B = R/I$,

$$\mathbf{E}^{(0)} : \quad \cdots \longrightarrow R \longrightarrow R \longrightarrow R \xrightarrow{\ f\ } B$$

with $E_n = R$ and the $d_i^n = s_j^n =$ identity homomorphism. Writing $\mathbf{K}(R,0) = \mathbf{E}^{(0)}$, it is easy to see that

$$\pi_0(\mathbf{K}(R,0)) \cong R \quad \text{and} \quad \pi_i(\mathbf{K}(R,0)) \cong 0 \quad \text{for} \quad i > 0.$$

Thus algebras are algebraic models of the 1-types of the 0-skeleton $\mathbf{E}^{(0)}$ of the 'step-by-step' construction.

4.2.2 2-Types

Again, given data for the 1-step of the construction of a free simplicial algebra which is

$$\mathbf{E}^{(1)}: \ldots R[s_0(X), s_1(X)] \underset{s_0, s_1}{\overset{d_0, d_1, d_2}{\rightrightarrows}} R[X] \underset{s_0}{\overset{d_0, d_1}{\rightrightarrows}} R \xrightarrow{\quad f \quad} R/I,$$

with

$$d_0^1(X_i) = 0, \quad d_1^1(X_i) = x_i \in \text{Ker} f, \quad s_0(r) = r \in R.$$

From the definition, there is an isomorphism

$$\pi_0(\mathbf{E}^{(1)}) \cong E_0^{(1)}/d_1^1(\text{Ker} d_0^1).$$

Consider the morphism

$$d_1^1 : \text{Ker} d_0^1 \longrightarrow R,$$

one readily obtains $\text{Im} d_1^1 = I$ and $E_0^{(1)} = R$. Thus

$$\pi_0(\mathbf{E}^{(1)}) \cong R/I.$$

Take a 1-truncation of a free simplicial algebra $\mathbf{E}^{(1)}$ as follows:

$$\ldots\ 0 \longrightarrow R[X]/P_1 \underset{s_0}{\overset{d_0, d_1}{\rightrightarrows}} R \xrightarrow{\quad f \quad} R/I.$$

Let $\mathbf{K}(B,1)$ denote this 1-truncated simplicial algebra. Using the proof of theorem 1.4.2, the corresponding free crossed module is

$$\partial_1 : R^+[X]/P_1 \longrightarrow R$$

By proposition 1.5.2, this becomes

$$R^n/\text{Im} d \longrightarrow R,$$

where d is the first Koszul differential. From the routine calculation above and corollary 3.4.4, there are the following isomorphisms

$$\pi_0(\mathbf{K}(B,1)) \cong B, \quad \pi_1(\mathbf{K}(B,1)) \cong \mathrm{Ker}(R^n/\mathrm{Im}d \longrightarrow R)$$

and

$$\pi_i(\mathbf{K}(B,1)) \cong 0 \quad \text{for } i > 1.$$

It then follows that free crossed modules are algebraic models of 2-types of the 1-skeleton of the free simplicial algebra.

4.2.3 3-Types

Suppose given the 2-skeleton $\mathbf{E}^{(2)}$ of the construction of a free simplicial algebra

$$\mathbf{E}^{(2)}: \ \ldots (R[s_0(X), s_1(X)])[Y]/P_2 \underset{s_0, s_1}{\overset{d_0, d_1, d_2}{\rightrightarrows}} R[X] \underset{s_0}{\overset{d_0, d_1}{\rightrightarrows}} R \xrightarrow{\ f\ } R/I.$$

As above, one gets the same homotopy modules, for $\mathbf{E}^{(2)}$, up to dimension 1:

$$\pi_0(\mathbf{E}^{(2)}) \cong B \quad \text{and} \quad \pi_1(\mathbf{E}^{(2)}) \cong \mathrm{Ker}(R^+[X]/P_1 \longrightarrow R).$$

By the remark in section 3.4, there is an isomorphism

$$\pi_2(\mathbf{E}^{(2)}) \cong \mathrm{Ker}(NE_2^{(2)}/\partial_3(NE_3^{(2)}) \longrightarrow E_1^{(2)})$$

Since $NE_2^{(2)} = (R[s_0(X), s_1(X)])^+[Y]$, the second homotopy module of the 2-skeleton looks like

$$\pi_2(\mathbf{E}^{(2)}) \cong \mathrm{Ker}((R[s_0(X), s_1(X)])^+[Y]/P_2 \longrightarrow R[X])$$

where P_2 is the second order Peiffer ideal.

Take a 2-truncation of a free simplicial algebra

$$\ldots\ 0 \longrightarrow (R[s_0(X), s_1(X)])[Y]/P_2 \underset{s_0, s_1}{\overset{d_0, d_1, d_2}{\rightrightarrows}} R[X] \underset{s_0}{\overset{d_0, d_1}{\rightrightarrows}} R \xrightarrow{\ f\ } R/I.$$

Let $\mathbf{Frtr}_2(\mathbf{E})$ denote this 2-truncated free simplicial algebra. Using theorem 4.1.7, the corresponding free 2-crossed module is

$$(R[s_0(X), s_1(X)])^+[Y]/P_2 \xrightarrow{\overline{\partial}_2} R^+[X] \xrightarrow{\partial_1} R.$$

From the above calculation, one has to get the subsequent isomorphisms:

$$\pi_0(\mathbf{Frtr_2}(\mathbf{E})) \cong B, \quad \pi_1(\mathbf{Frtr_2}(\mathbf{E})) \cong \mathrm{Ker}(R^+[X]/P_1 \longrightarrow R)$$

and

$$\pi_2(\mathbf{Frtr_2}(\mathbf{E})) \cong \mathrm{Ker}((R[s_0(X), s_1(X)])^+[Y]/P_2 \longrightarrow R[X])$$

and finally

$$\pi_i(\mathbf{Frtr_2}(\mathbf{E})) \cong 0 \quad \text{for } i > 2.$$

Hence free 2-crossed modules of algebras are algebraic models of 3-types of the 2-skeleton of the free simplicial algebra.

CHAPTER 5

CROSSED SQUARES AND CROSSED N-CUBES

5.1 CROSSED SQUARES

A 2-dimensional version of a crossed module, called a crossed square, was defined by D.Guin-Waléry and J.L.Loday, [25] in 1981. The commutative algebra analogue has been studied by G.J.Ellis [18]. In this section, we show how higher order Peiffer identities are present in the definition of a crossed square.

Definition 5.1.1 A crossed square of algebras *is a commutative diagram of commutative algebras*

$$\begin{array}{ccc} B & \xrightarrow{\ \delta\ } & D \\ {\scriptstyle \delta'}\big\downarrow & & \big\downarrow{\scriptstyle \partial'} \\ C & \xrightarrow[\ \partial\ ]{} & R \end{array}$$

together with actions of R on B, C and D. There are thus commutative actions (see chapter 1) of C on B and D via ∂, and D acts on B and C via ∂' and a function $h : C \times D \to B$ such that, for all $c, c' \in C,\ d, d' \in D,\ r \in R,\ b \in B,\ k \in \mathbf{k}$;

1. *each of the maps $\delta,\ \delta',\ \partial,\ \partial'$ and the composite $\partial'\delta = \partial\delta'$ are crossed modules,*

2. *the maps* δ, δ' *preserve the action of* R,

3. $kh(c,d) = h(kc,d) = h(c,kd)$,

4. $h(c + c', d) = h(c,d) + h(c',d)$,

5. $h(c, d + d') = h(c,d) + h(c,d')$,

6. $r \cdot h(c,d) = h(r \cdot c, d) = h(c, r \cdot d)$,

7. $\delta h(c,d) = c \cdot d$,

8. $\delta' h(c,d) = d \cdot c$,

9. $h(c, \delta b) = c \cdot b$,

10. $h(\delta' b, d) = d \cdot b$.

A morphism of crossed squares $\Phi : (B,C,D,R) \to (B',C',D',R')$, consists of homomorphisms

$$\Phi_B : B \to B' \qquad \Phi_C : C \to C'$$
$$\Phi_D : D \to D' \qquad \Phi_R : R \to R',$$

such that the cube of homomorphisms is commutative;

$$\Phi_B h(c,d) = h(\Phi_C c, \Phi_D d) \qquad \text{with } c \in C,\ d \in D,$$

and each of homomorphisms Φ_B, Φ_C, Φ_D is Φ_R-equivariant. The category of crossed squares will be denoted, $\mathbf{Crs^2}$.

Example 5.1.2 *Let* I_1, I_2 *be ideals of the* $\mathbf{k}$*-algebra* R*. The commutative diagram of inclusions;*

$$\begin{array}{ccc} I_1 \cap I_2 & \xrightarrow{inc.} & I_2 \\ {\scriptstyle inc.}\downarrow & & \downarrow{\scriptstyle inc.} \\ I_1 & \xrightarrow[inc.]{} & R \end{array}$$

together with the actions of R *on* I_1, I_2 *and* $I_1 \cap I_2$ *given by multiplication and the function*

$$\begin{array}{rccc} h: & I_1 \times I_2 & \longrightarrow & I_1 \cap I_2 \\ & (i_1, i_2) & \longmapsto & i_1 i_2, \end{array}$$

is a crossed square as is easily checked.

Proposition 5.1.3 *Let* $\mathbf{E}$ *be a simplicial algebra with simplicial ideals* $\mathbf{I_1}$ *and* $\mathbf{I_2}$. *Then a square*

$$\begin{array}{ccc} \boldsymbol{I_1} \cap \boldsymbol{I_2} & \longrightarrow & \boldsymbol{I_2} \\ \downarrow & & \downarrow \\ \boldsymbol{I_1} & \longrightarrow & \boldsymbol{E} \end{array}$$

induces a crossed square

$$\begin{array}{ccc} \pi_0(\boldsymbol{I_1} \cap \boldsymbol{I_2}) & \longrightarrow & \pi_0(\boldsymbol{I_2}) \\ \downarrow & & \downarrow \\ \pi_0(\boldsymbol{I_1}) & \longrightarrow & \pi_0(\boldsymbol{E}). \end{array}$$

Proof: The h-function

$$h : \pi_0(\mathbf{I_1}) \times \pi_0(\mathbf{I_2}) \longrightarrow \pi_0(\mathbf{I_1} \cap \mathbf{I_2})$$

is given by

$$h([a],[b]) = [a][b] = [ab]$$

for all $[a] \in \pi_0(\mathbf{I_1})$, $[b] \in \pi_0(\mathbf{I_2})$. It follows from lemma 1.3.5 that the above diagram is a crossed square. □

Here again the generating elements of $\partial_3 NE_3$ pays off the following result.

Proposition 5.1.4 *Let* $\mathbf{E}$ *be a simplicial algebra. Then the following diagram*

$$\begin{array}{ccc} NE_2/\partial_3 NE_3 & \xrightarrow{\delta} & NE_1 \\ \delta' \downarrow & & \downarrow \partial' \\ \overline{NE_1} & \xrightarrow[\partial]{} & E_1 \end{array}$$

is a crossed square. Here NE_1 =Kerd_0^1 *and* $\overline{NE}_1$ =Kerd_1^1.

Proof: Since E_1 acts on $NE_2/\partial_3 NE_3$, $\overline{NE}_1$ and NE_1, there are actions of $\overline{NE}_1$ on $NE_2/\partial_3 NE_3$ and NE_1 via ∂, and NE_1 acts on $NE_2/\partial_3 NE_3$ and $\overline{NE}_1$ via ∂'. As ∂ and ∂' are inclusions, all actions can be given by multiplication. The h-map is

$$\begin{array}{ccc} NE_1 \times \overline{NE}_1 & \longrightarrow & NE_2/\partial_3 NE_3 \\ (x, \overline{y}) & \longmapsto & h(x,\overline{y}) = s_1x(s_1y - s_0y) + \partial_3 NE_3, \end{array}$$

which is bilinear. Here x and y are in NE_1 as there exists a bijection between NE_1 and $\overline{NE}_1$ (by lemma 2.3.1). We next verify the crossed square axioms.

Axiom (1) : the two morphisms with codomain E_1 are inclusions of ideal subalgebras hence are crossed modules; the two with domain $NE_2/\partial_3 NE_3$ are induced by d_2. Thus ∂ and ∂' can be easily shown to be crossed modules as they are inclusions. In the following we just verify that the composite $\partial'\delta$ is a crossed module.

If $a + \partial_3 NE_3$, $b + \partial_3 NE_3 \in NE_2/\partial_3 NE_3$, it then follows that

$$\begin{array}{rcll} (\partial'\delta)(a + \partial_3 NE_3)\cdot(b + \partial_3 NE_3) & = & \partial'\delta(a)\cdot b + \partial_3 NE_3 & \\ & = & \partial' s_1 d_2(a) b + \partial_3 NE_3 & \text{by the action} \\ & = & s_1 d_2(a) b + \partial_3 NE_3 & \text{by } \partial' \text{ inclusion} \\ & \equiv & ab + \partial_3 NE_3 & \mod \partial_3 NE_3 \\ & = & (a + \partial_3 NE_3)(b + \partial_3 NE_3) & \end{array}$$

As it is seen, the verifying of $\partial'\delta$ is more or less identical the proof that

$$NE_1/\partial_2 NE_2 \longrightarrow NE_0$$

is a crossed module (see section 3.4). Likewise δ, δ' and $\partial\delta'$ are crossed modules.

Axioms (2), (3) are obvious. Since the h-map is bilinear that implies axioms (4) and (5) hold. Axiom (6) is also easily checked (see $PL5$ of proposition 4.1.3).

Axiom (7) :

$$\begin{array}{rcl} \delta h(x,y) & = & d_2(s_1x(s_1y - s_0y)) \\ & = & xy - xs_0d_1y \\ & \equiv & xy \quad \text{since } \partial_2(NE_2) \\ & = & x\cdot y. \end{array}$$

as $C_{(1)(0)}(x\otimes y) = s_1x(s_0y - s_1y)$ (see section 2.4.1) and $d_2C_{(1)(0)}(x\otimes y) = x(s_0d_1y - y)$. Similarly the following axiom, (8), $\delta' h(x,y) = -y\cdot x$ is satisfied.

Axiom (9) : For $z \in NE_2/\partial_3 NE_3$,

$$\begin{aligned} h(x,\delta z) &= s_1x(s_1d_2z - s_0d_2z) \\ &= s_1(x)s_1d_2z - s_1(x)s_0d_2z \\ &\equiv s_1(x)z \qquad \text{mod } \partial_3 NE_3 \\ &= x \cdot z. \end{aligned}$$

(See for details the b of $PL4$ of proposition 4.1.3.)

Axiom (10) :

$$\begin{aligned} h(\delta' z, y) &= s_1d_2z(s_1y - s_0y) \\ &= s_1d_2(z)s_1y - s_1d_2(z)s_0y \\ &\equiv -(s_1y - s_0y)z \qquad \text{mod } \partial_3 NE_3 \\ &= -(s_1 - s_0)(y)z \\ &= -y \cdot z \quad \text{by the definition of the action.} \end{aligned}$$

(See for details the a of $PL4$ of proposition 4.1.3). □

This result presents the following functor

$$\mathbf{M_2} : \mathbf{SimpAlg} \longrightarrow \mathbf{Crs_2}$$

5.2 Crossed n-cubes

Crossed n-cubes in algebraic settings such as commutative algebras, Jordan algebras, Lie algebras have been defined by G.J.Ellis [19]. Here we recall from [19] the case of crossed n-cube of commutative algebras and give some examples.

Definition 5.2.1 A crossed n-cube of commutative algebras *is a family of commutative algebras, M_A for $A \subseteq <n> = \{1,\ldots,n\}$ together with homomorphisms $\mu_i : M_A \to M_{A-\{i\}}$ for $i \in <n>$ and for $A,B \subseteq <n>$, functions*

$$h : M_A \times M_B \longrightarrow M_{A\cup B}$$

such that for all $k \in \mathbf{k},\ a, a' \in M_A,\ b, b' \in M_B,\ c \in M_C,\ i, j \in <n>$ *and* $A \subseteq B$

$$
\begin{array}{lrcl}
1) & \mu_i a & = & a \quad \text{if } i \notin A \\
2) & \mu_i \mu_j a & = & \mu_j \mu_i a \\
3) & \mu_i h(a,b) & = & h(\mu_i a, \mu_i b) \\
4) & h(a,b) & = & h(\mu_i a, b) = h(a, \mu_i b) \quad \text{if } i \in A \cap B \\
5) & h(a,a') & = & aa' \\
6) & h(a,b) & = & h(b,a) \\
7) & h(a+a',b) & = & h(a,b) + h(a',b) \\
8) & h(a,b+b') & = & h(a,b) + h(a,b') \\
9) & k \cdot h(a,b) & = & h(k \cdot a, b) = h(a, k \cdot b) \\
10) & h(h(a,b),c) & = & h(a,h(b,c)) = h(b,h(b,c)).
\end{array}
$$

A morphism of crossed n-cubes is defined in the obvious way: It is a family of commutative algebra homomorphisms, for $A \subseteq <n>$

$$f_A : M_A \longrightarrow M'_A$$

commuting with the μ_i's and h's. We thus obtain a category of crossed n-cubes denoted by $\mathbf{Crs^n}$.

Example 5.2.2 *For* $n = 1$, *a crossed 1-cube is the same as a crossed module. For* $n = 2$, *one has a crossed square as above:*

$$
\begin{array}{ccc}
M_{<2>} & \xrightarrow{\mu_2} & M_{\{1\}} \\
\mu_1 \downarrow & & \downarrow \mu_1 \\
M_{\{2\}} & \xrightarrow[\mu_2]{} & M_{\emptyset}.
\end{array}
$$

Each μ_i *is a crossed module as is* $\mu_1 \mu_2$. *The h-functions give actions and a function*

$$h : M_{\{1\}} \times M_{\{2\}} \longrightarrow M_{<2>}.$$

The maps μ_2 *(or* μ_1*) also define a map of crossed modules. In fact a crossed square can be thought of as a crossed module in the category of crossed modules.*

By *an ideal* $(n+1)$*-ad* will be meant an algebra with n-ideals (possibly with repeats).

Example 5.2.3 *Let R be an algebra with ideals* $I_1, \ldots, I_n$ *of R. Let*

$$M_A = \bigcap\{I_i : i \in A\} \quad and \quad M_{\emptyset} = R$$

with $A \subseteq < n >$ *. If* $i \in < n >$*, then* M_A *is the ideals of* $M_{A-\{i\}}$*. Define*

$$\mu_i : M_A \longrightarrow M_{A-\{i\}}$$

to be the inclusion. If $A, B \subseteq < n >$*, then* $M_{A\cup B} = M_A \cap M_B$*, let*

$$\begin{array}{rccc} h: & M_A \times M_B & \longrightarrow & M_{A\cup B} \\ & (a,b) & \longmapsto & ab \end{array}$$

as $M_A M_B \subseteq M_A \cap M_B$*, where* $a \in M_A,\ b \in M_B$*. Then*

$$\{M_A :\ A \subseteq < n >,\ \mu_i,\ h\}$$

is a crossed n-cube, called the inclusion crossed n-cube *given by the ideal* $(n+1)$*-ad of commutative algebras* $(R;\ I_1, \ldots, I_n)$*.*

Proposition 5.2.4 *Let* $(\mathbf{E};\ I_1, \ldots, I_n)$ *be a simplicial ideal* $(n+1)$*-ad of algebras and define for* $A \subseteq < n >$

$$M_A = \pi_0(\bigcap_{i\in A} I_i)$$

with homomorphisms $\mu_i : M_A \to M_{A-\{i\}}$ *and h-maps induced by the corresponding maps in the simplicial inclusion crossed n-cube, constructed by applying the previous example to each level. Then* $\{M_A :\ A \subseteq < n >,\ \mu_i,\ h\}$ *is a crossed n-cube.*

Proof: As the proof is the obvious extension to crossed n-cubes of the proof for $n = 2$ above (proposition 5.1.3), it has been omitted. □

Up to isomorphism, all crossed n-cubes arise in this way. In fact any crossed n-cube can be realised (up to isomorphism) as a π_0 of a simplicial inclusion crossed n-cube coming from a simplicial ideal (n+1)-ad in which π_0 is a non-trivial homotopy module.

5.3 From SimpAlg. to Crs^n

In 1991, T.Porter [38] described the functor from the category of simplicial groups to that of crossed n-cubes of groups.

In this section, we adapt his description to give an obvious analogue of this functor for the algebra case. The functor here constructed is defined using the décalage functor studied by Illusie [26] and Duskin [16] and is a π_0-image of a functor taking values in a category of simplicial ideal (n+1)-ads. The décalage functor forgets the last face operators at each level of a simplicial algebra **E** and moves everything down one level. It is denoted by Dec. Thus

$$(\mathrm{Dec}E)_n = E_{n+1}.$$

The last degeneracy of **E** yields a contraction of $\mathrm{Dec}^1\mathbf{E}$ as an augmented simplicial algebra,

$$\mathrm{Dec}^1\mathbf{E} \simeq \mathbf{K}(E_0, 0),$$

by an explicit natural homotopy equivalence (c.f. [16]). The last face map will be denoted

$$\mathrm{Dec}^1\mathbf{E} \longrightarrow \mathbf{E}.$$

Iterating the Dec construction gives an augmented bisimplicial algebra

$$[\ldots\ \mathrm{Dec}^3\mathbf{E} \rightrightarrows\!\!\!\rightarrow \mathrm{Dec}^2\mathbf{E} \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} \mathrm{Dec}^1\mathbf{E}]$$

which in expanded form is the total décalage of **E:**

$$[\ldots\ \mathrm{Dec}^3\mathbf{E} \rightrightarrows\!\!\!\rightarrow \mathrm{Dec}^2\mathbf{E} \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} \mathrm{Dec}^1\mathbf{E}] \xrightarrow{\ \delta_0\ } \mathbf{E}.$$

(see [16] or [26] for details). The maps from $\mathrm{Dec}^i\mathbf{E}$ to $\mathrm{Dec}^{i-1}\mathbf{E}$ coming from the ith last face maps will be labelled $\delta_0, \ldots, \delta_{i-1}$ so that $\delta_0 = d_{last}$, $\delta_1 = d_{last\ \ but\, one}$ and so on.

For a simplicial algebra **E** and a given n, we write $\mathbf{M}(\mathbf{E}, n)$ for a crossed n-cube, arising as a functor

$$\mathbf{M(-, n) : SimpAlg \longrightarrow Crs^n}.$$

The following data determines a crossed n-cube of algebras:

Theorem 5.3.1 *If* $\mathbf{E}$ *be a simplicial algebra, then the crossed n-cube* $\mathbf{M}(\mathbf{E},n)$ *is determined by:*

(i) for $A \subseteq < n >$,

$$\mathbf{M}(\mathbf{E},n)_A = \frac{\bigcap_{j\in A}\mathrm{Ker}d^n_{j-1}}{d^{n+1}_{n+1}(\mathrm{Ker}d^{n+1}_0 \cap \{\bigcap_{j\in A}\mathrm{Ker}d^{n+1}_j\});}$$

(ii) the inclusion

$$\bigcap_{j\in A}\mathrm{Ker}d^n_{j-1} \longrightarrow \bigcap_{j\in A-\{i\}}\mathrm{Ker}d^n_{j-1}$$

induces the morphism

$$\mu_i : \mathbf{M}(\mathbf{E},n)_A \longrightarrow \mathbf{M}(\mathbf{E},n)_{A-\{i\}};$$

(iii) the functions, for $A, B \subseteq < n >$,

$$h : \mathbf{M}(\mathbf{E},n)_A \times \mathbf{M}(\mathbf{E},n)_B \longrightarrow \mathbf{M}(\mathbf{E},n)_{A\cup B}$$

given by

$$h(\bar{x},\bar{y}) = \bar{x}\bar{y},$$

where an element of $\mathbf{M}(\mathbf{E},n)_A$ *is denoted by* $\bar{x}$ *with* $x \in \bigcap_{j\in A}\mathrm{Ker}d^n_{j-1}$.

Proof: For each simplicial algebra, $\mathbf{E}$, we start by looking at the canonical augmentation map

$$\delta_0 : \mathrm{Dec}^1\mathbf{E} \longrightarrow \mathbf{E},$$

which has kernel the simplicial algebra, $\mathrm{Ker}d_{last}$ used above. Then take the simplicial inclusion crossed module

$$\mathrm{Ker}\delta_0 \longrightarrow \mathrm{Dec}^1\mathbf{E}$$

to be $\mathscr{M}(\mathbf{E},1)$ defining thus a functor

$$\mathscr{M}(\quad,1) : \mathbf{SimpAlg} \longrightarrow \mathbf{Simp(IncCrs^1)}.$$

Then it is easy to show that

$$\pi_0(\mathrm{Ker}\delta_0) \longrightarrow \pi_0(\mathrm{Dec}^1\mathbf{E})$$

is precisely $\mathbf{M}(\mathbf{E},1)$. The higher order analogues $\mathscr{M}(\quad,1)$ are as follows: For each simplicial algebra, $\mathbf{E}$, there is a functorial short exact sequence

$$\mathrm{Ker}\delta_0 \longrightarrow \mathrm{Dec}^1\mathbf{E} \longrightarrow \mathbf{E}.$$

This corresponds to the 0-skeleton of the total décalage of **E**, i.e.

$$[\ldots \ \mathrm{Dec}^3\mathbf{E} \ \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \ \mathrm{Dec}^2\mathbf{E} \ \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} \ \mathrm{Dec}^1\mathbf{E}] \ \xrightarrow{\delta_0} \ \mathbf{E}.$$

For $n = 2$, the 1-skeleton of that total décalage gives a commutative diagram

$$\begin{array}{ccc} \mathrm{Dec}^2\mathbf{E} & \xrightarrow{\delta_0} & \mathrm{Dec}^1\mathbf{E} \\ {\scriptstyle \delta_1}\big\downarrow & & \big\downarrow{\scriptstyle \delta_0} \\ \mathrm{Dec}^1\mathbf{E} & \xrightarrow{\delta_0} & \mathbf{E}. \end{array}$$

Here δ_1 is d^n_{n-1} in dimension n whilst δ_0 is d^n_n. Forming the square of kernels gives

$$\begin{array}{ccc} \mathrm{Ker}\delta_0 \cap \mathrm{Ker}\delta_1 & \longrightarrow & \mathrm{Ker}\delta_1 \\ \big\downarrow & & \big\downarrow \\ \mathrm{Ker}\delta_0 & \longrightarrow & \mathrm{Dec}^2\mathbf{E}. \end{array}$$

Again, π_0 of this gives $\mathbf{M}(\mathbf{E}, 2)$. In general, we use the $(n-1)$-skeleton of the total décalage to form an n-cube. Thus a simplicial inclusion crossed n-cube continuing this n-times given the simplicial inclusion crossed n-cube corresponding to the simplicial ideal $(n+1)$-ad ($\mathrm{Dec}^n E$; $\mathrm{Ker}\delta_{n+1}, \ldots, \mathrm{Ker}\delta_0$). This simplicial inclusion n-cube will be denoted by $\mathscr{M}(\mathbf{E}, n)$, and its associated crossed n-cube by

$$\pi_0(\mathscr{M}(\mathbf{E}, n)) = \mathbf{M}(\mathbf{E}, n).$$

This follows from Proposition 5.2.4 that the description of π_0 as the H_0 of the Moore complex. So the formula in (i) is somewhat simple by the definition of H_0. □

The following lemma is proved by Porter for the group case. His proof adapts easily but is included for completeness.

Lemma 5.3.2 *If* **E** *is a simplicial algebra with* $A \subseteq < n >$, $A \neq < n >$, *then*

$$d_n\Big(\bigcap_{i \in A} \mathrm{Ker} d^n_i\Big) = \bigcap_{i \in A} \mathrm{Ker} d^{n-1}_{i-1}.$$

Proof: If $i \in A$, then

$$d_n(\bigcap_{i\in A} \mathrm{Ker} d_i^n) \subseteq \bigcap_{i\in A} \mathrm{Ker} d_{i-1}^{n-1},$$

since $d_{i-1}d_n = d_{n-1}d_{i-1}$.

Conversely, we suppose that x is an element in $\bigcap_{i\in A} \mathrm{Ker} d_{i-1}^{n-1}$ and consider the element

$$y = s_n x - s_{n-1}x + \ldots + (-1)^{n-k} s_k x = \sum_{i=0}^{n-k} (-1)^{i+1} s_{i+k} x,$$

where k is the first integer in $< n > \backslash A$. Then

$$d_n y = x \quad \text{and} \quad d_i y = 0 \quad \text{for all} \quad i \in A$$

and hence $y \in \bigcap_{i\in A} \mathrm{Ker} d_i^n$ implies $x \in d_n(\bigcap_{i\in A} \mathrm{Ker} d_i^n)$ as required. □

This lemma gives the following proposition:

Proposition 5.3.3 *If* **E** *is a simplicial algebra, then*

i) for $A \subseteq < n >$, $A \neq < n >$,

$$\mathbf{M}(\mathbf{E}, n)_A \cong \bigcap_{i\in A} \mathrm{Ker} d_{i-1}^{n-1}$$

so that in particular, $\mathbf{M}(\mathbf{E}., n)_\emptyset \cong E_{n-1}$*; in every case the isomorphism is induced by* d_0*,*

ii) if $A \neq < n >$ and $i \in < n >$,

$$\mu_i : \mathbf{M}(\mathbf{E}., n)_A \longrightarrow \mathbf{M}(\mathbf{E}., n)_{A\backslash\{i\}}$$

is the inclusion of an ideal,

iii) for $j \in < n >$,

$$\mu_j : \mathbf{M}(\mathbf{E}., n)_{<n>} \longrightarrow \bigcap_{i\neq j} \mathrm{Ker} d_i^{n+1}$$

is induced by d_n.

Proof: By theorem 5.3.1 and the previous lemma, one can obtain, for $A \neq < n >$,

$$\begin{aligned}\mathbf{M}(\mathbf{E}., n)_A &= \frac{\bigcap_{i\in A} \mathrm{Ker} d_{i-1}^n}{d_{n+1}(\mathrm{Ker} d_0^{n+1} \cap \{\bigcap_{i\in A} \mathrm{Ker} d_i^{n+1}\})} \\ &= \frac{\bigcap_{i\in A} \mathrm{Ker} d_{i-1}^n}{\mathrm{Ker} d_0^n \cap (\bigcap_{i\in A} \mathrm{Ker} d_{i-1}^n).}\end{aligned}$$

The epimorphism $d_0 : E_n \to E_{n-1}$, which is $d_0 s_0 = \mathrm{id}$, can be restricted to an epimorphism

$$\bigcap_{i\in A} \mathrm{Ker} d^n_{i-1} \longrightarrow \bigcap_{i\in A} \mathrm{Ker} d^{n-1}_{i-1},$$

by lemma 5.3.2. It follows then that

$$\mathrm{Ker}(\bigcap_{i\in A} \mathrm{Ker} d^n_{i-1} \xrightarrow{d_0} \bigcap_{i\in A} \mathrm{Ker} d^{n-1}_{i-1}) = \mathrm{Ker} d^n_0 \cap (\bigcap_{i\in A} \mathrm{Ker} d^n_{i-1}).$$

which completes the proof of (i).

(ii) and (iii) are now consequences. □

Remark 5.3.4 *1) For $n = 0$,*

$$\begin{aligned} \mathbf{M}(\mathbf{E}.,0) &= E_0/d_1(\mathrm{Ker} d_0) \\ &\cong \pi_0(\mathbf{E}) \\ &= H_0(\mathbf{E}). \end{aligned}$$

2) For $n = 1$, $\mathbf{M}(\mathbf{E}., n)$ is the crossed module

$$\mu_1 : \mathrm{Ker} d^1_0 / d^2_2(NE_2) \longrightarrow E_1/d^2_2(\mathrm{Ker} d^2_0).$$

Since $d^2_2(NE_2) = \mathrm{Ker} d^1_0 \mathrm{Ker} d^1_1$, this implies

$$\mu : NE_1/\mathrm{Ker} d^1_0 \mathrm{Ker} d^1_1 \longrightarrow E_0.$$

3) For $n = 2$, $\mathbf{M}(\mathbf{E}., n)$ is

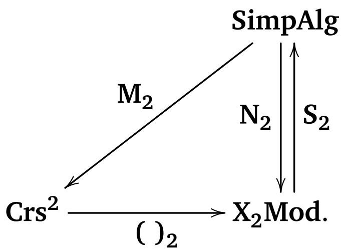
$$\begin{array}{ccc} Kerd^2_0 \cap Kerd^2_1/d^3_3(Kerd^3_0 \cap Kerd^3_1 \cap Kerd^3_2) & \xrightarrow{\mu_2} & Kerd^2_0/d^3_3(Kerd^3_0 \cap Kerd^3_1) \\ \downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle \mu_1} \\ Kerd^2_1/d^3_3(Kerd^3_0 \cap Kerd^3_2) & \xrightarrow{\mu_2} & E_2/d^3_3(Kerd^3_0). \end{array}$$

By proposition 5.3.3, this is isomorphic to

$$\begin{array}{ccc} NE_2/d^3_3(NE_3) & \xrightarrow{\mu_2} & Kerd^1_0 \\ \downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle \mu_1} \\ Kerd^1_1 & \xrightarrow{\mu_2} & E_1. \end{array}$$

is the crossed square in which proposition 5.1.4 confirms this result.

D.Conduché's unpublished work determines that there exists an equivalence (up to homotopy) between the category of crossed squares of groups and that of 2-crossed modules of groups. We think that this result is true for the commutative algebra case, but we have not proved it. As for that, the situation in chapter 4 and chapter 5 may be abridged in the following diagram



5.4 Free Crossed Squares

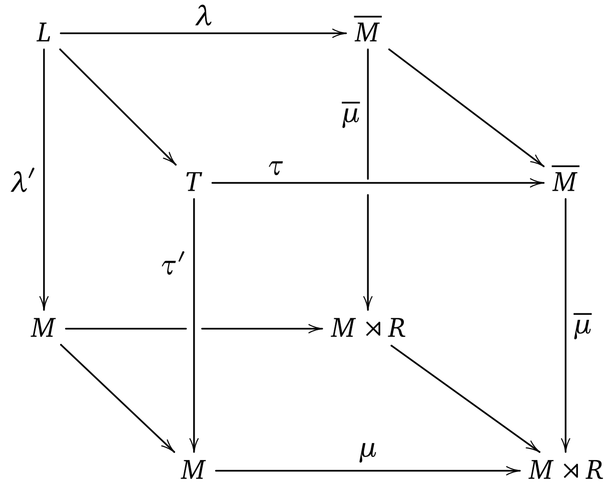
G.Ellis, [21], in 1993 presented the notion of a free crossed square for the case of groups. In this section, we introduce the commutative algebra version of this definition and give a construction of a free crossed square by using the second order Peiffer elements and the 2-skeleton of step-by-step construction of a free simplicial algebra.

We firstly define the free crossed square on a pair of function (f_2, f_3).

Definition 5.4.1 *Let $(L, M, \bar{M}, M \rtimes R)$ be a crossed square. Suppose given a function $f_2 : \mathbf{S_2} \to R$, from a set $\mathbf{S_2}$ to an algebra R. Let $\partial : M \to R$ be the free pre-crossed module on f_2. Assume given a function from a set $\mathbf{S_3}$ to M, namely $f_3 : \mathbf{S_3} \to M$. Then*

$(L, M, \bar{M}, M \rtimes R)$ is said to be a free crossed square on a pair of functions (f_2, f_3) if for any crossed square $(T, M, \bar{M}, M \rtimes R)$ and function $\nu : \mathbf{S_3} \to T$, there is a unique morphism ϕ of

crossed squares:



such that $\tau\nu = \lambda$.

We denote such a free crossed square of algebras by $(L, M, \bar{M}, M \rtimes R)$. The category of free crossed squares will be denoted, $\mathbf{FrCrs}^2$.

We will present a precise description of the construction of a free crossed square by using the third property of remark 5.3.4 and the second order Peiffer ideal. To do this we need to recall the 2-dimensional construction for a free simplicial algebra in Ellis's notation. This 2-dimensional form can be pictured by the diagram

$$\mathbf{E}^{(2)} : \ldots (R[s_0(\mathbf{S_2}), s_1(\mathbf{S_2})])[\mathbf{S_3}] \underset{s_0, s_1}{\overset{d_0, d_1, d_2}{\rightrightarrows}} R[\mathbf{S_2}] \underset{s_0}{\overset{d_0, d_1}{\rightrightarrows}} R \xrightarrow{\ f\ } B$$

with the simplicial identities given as before. Here $\mathbf{S_2} = \{S_1, \ldots, S_n\}$ and $\mathbf{S_3} = \{S'_1, \ldots, S'_m\}$ are finite sets and take R to be the algebra and $B = R/(t_1, \ldots, t_n)$ as an R-algebra.

Theorem 5.4.2 *A free crossed square* $(L, M, \bar{M}, M \rtimes R)$ *exists on the 2-dimensional construction data.*

Proof: Suppose given the 2-dimensional construction data for a free simplicial algebra and a function

$$f_2 : \mathbf{S_2} \longrightarrow R.$$

From the routine calculation of lemma 3.4.1, we have

$$M = \mathrm{Ker} d_0^1 = NE_1^{(2)} = R^+[\mathbf{S_2}] = (S_1, \ldots, S_n),$$

where explicit elements of $R^+[\mathbf{S_2}]$ are of the form

$$\sum_{\alpha\in\Lambda} r_\alpha S_1^{i_1}\dots S_n^{i_n}$$

with Λ a set of multi-indices and some $r_{i_1,\dots,i_n}\in R$. It is easy to see that

$$\partial_1 : R^+[\mathbf{S_2}] \longrightarrow R$$

is a free pre-crossed module on f_2, where $\partial_1 = d_1$. Since lemma 2.2.2, one form the semidirect product as follows

$$\begin{aligned} R[\mathbf{S_2}] = E_1^{(2)} &\cong \operatorname{Ker} d_0^1 \rtimes s_0 E_0^{(2)}, \\ &= R^+[\mathbf{S_2}] \rtimes s_0 R, \\ &= M \rtimes s_0 R, \\ &= M \rtimes R \quad \text{by } \ s_0(r) = r, \ \text{ for all } \ r\in R, \end{aligned}$$

and so $E_1^{(2)} \cong M \rtimes R$. Then take the canonical inclusion

$$R^+[\mathbf{S_2}] \longrightarrow R^+[\mathbf{S_2}] \rtimes s_0 R$$

given by $S_i \longmapsto (S_i, 0)$. The other ideal of $R[\mathbf{S_2}]$ is obtained from $\operatorname{Ker} d_0^1 = R^+[\mathbf{S_2}]$, namely

$$\bar{M} = \operatorname{Ker} d_1^1 = \overline{R^+[\mathbf{S_2}]} = (S_1 - t_1, \dots, S_n - t_n),$$

where precise elements of $\overline{R^+[\mathbf{S_2}]}$ are of the form

$$\sum_{\alpha\in\Lambda} r_\alpha \left((S_1^{i_1}\dots S_n^{i_n}) - (t_1^{i_1}\dots t_n^{i_n}) \right).$$

In other words, if $m = S_i$, then $\bar{m} = S_i - s_0 d_1(S_i) = S_i - t_i$ (by lemma 2.2.2), with $t_i \in R$. So we denote the elements $\bar{m} = (S_i - \partial_1 S_i)$. Assume given a function

$$f_3 : \mathbf{S_3} \longrightarrow R^+[\mathbf{S_2}]$$

with $\operatorname{Im} f_3 \subseteq \operatorname{Ker}\partial_1$. There is then a corresponding function :

$$\begin{aligned} \bar{f}_3 : \quad \mathbf{S_3} &\longrightarrow \bar{M} \\ y &\longmapsto (f_3 y, 0). \end{aligned}$$

Let $(T, M, \bar{M}, M \rtimes R)$ be any crossed square and let function $\nu : \mathbf{S_3} \to T$. We will show that the free crossed square may be pictured by

$$\begin{array}{ccc} L & \xrightarrow{\partial'} & \overline{R^+[\mathbf{S_2}]} \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\lambda'} \\ R^+[\mathbf{S_2}] & \xrightarrow[\lambda]{} & R^+[\mathbf{S_2}] \rtimes R. \end{array}$$

In other words,

$$\begin{array}{ccc} L & \xrightarrow{\partial'} & \mathrm{Ker}d_0^1 \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\lambda'} \\ \mathrm{Ker}d_1^1 & \xrightarrow[\lambda]{} & E_1^{(2)} \cong \mathrm{Ker}d_1 \rtimes s_0 E_0^{(2)}. \end{array}$$

Taking $L = NE_2^{(2)}/\partial_3(NE_3^{(2)})$ which gives the crossed 2-cube $\mathbf{M(E}^{(2)}, 2\ \mathbf{)}$, namely

$$\begin{array}{ccc} NE_2^{(2)}/\partial_3(NE_3^{(2)}) & \xrightarrow{\mu_2} & \mathrm{Ker}d_0^1 \\ {\scriptstyle\mu_1}\downarrow & & \downarrow{\scriptstyle\mu_1} \\ \mathrm{Ker}d_1^1 & \xrightarrow{\mu_2} & E_1^{(2)}. \end{array}$$

Next investigate L. As for the above notations in the 2-skeleton of the free simplicial algebra and by proposition 2.4.1, $\partial_3(NE_3^{(2)})$ is generated by elements of the form

$$(s_1 s_0 d_1 S_i - s_0 S_i) S'_j,$$
$$(s_0 S_i - s_1 S_i)(s_1 d_2 S'_j - S'_j),$$
$$s_1 S_i (s_0 d_2 S'_j - s_1 d_2 S'_j + S'_j),$$

and for S'_i, $S'_j \in NE_2$,

$$S'_i(s_1 d_2 S'_j - S'_j),$$
$$S'_i(S'_j + s_0 d_2 S'_j - s_1 d_2 S'_j),$$
$$(s_0 d_2 S'_i - s_1 d_2 S'_i + S'_i)(s_1 d_2 S'_j - S'_j),$$

which are the second order Peiffer elements, where $S_i \in NE_1 = \mathrm{Ker}d_0 = R^+[\mathbf{S_2}]$ and $S'_i \in NE_2 = \mathrm{Ker}d_0 \cap \mathrm{Ker}d_1 = (R[s_0(\mathbf{S_2}),\ s_1(\mathbf{S_2})])^+[\mathbf{S_3}]$.

The above diagram can be given by

$$\begin{array}{ccc} (R[s_0(\mathbf{S_2}), s_1(\mathbf{S_2})])^+[(\mathbf{S_3})]/P_2 & \xrightarrow{\partial'} & \overline{R^+[\mathbf{S_2}]} \quad (*) \\ \Big\downarrow{\scriptstyle \partial} & & \Big\downarrow{\scriptstyle \lambda'} \\ R^+[\mathbf{S_2}] & \xrightarrow[\lambda]{} & R^+[\mathbf{S_2}] \rtimes R \end{array}$$

here P_2 is the second order Peiffer ideal in $(R[s_0(\mathbf{S_2}),\ s_1(\mathbf{S_2})])^+[\mathbf{S_3}]$. Hence there exists a unique morphism

$$\phi : (L, M, \bar{M}, M \rtimes R) \longrightarrow (T, M, \bar{M}, M \rtimes R)$$

is given by

$$\phi(S'_i + P_2) = \nu(S'_i)$$

such that $\tau\nu = \partial'$, where $\tau : T \to \bar{M}$ is a morphism. Thus diagram $(*)$ is the desired free crossed square on the 2-dimensional construction data. In a similar way to proposition 5.1.4, the free crossed square axioms may be verified. $\Box$

We have thus showed that the construction of free crossed square corresponds to the crossed 2-cubes. Therefore we can say that $\mathbf{M}(\mathbf{E}^{(2)}, 2\ \mathbf{)}$ is a free crossed square.

By a 'step-by-step' construction of a free simplicial algebra, there are simplicial inclusions

$$\mathbf{E}^{(0)} \subseteq \mathbf{E}^{(1)} \subseteq \mathbf{E}^{(2)} \ldots$$

Considering the functor, described the previous section, $\mathbf{M}(\mathbf{E}, n)$ from the category of simplicial algebras to that of crossed n-cubes which gives the ensuing inclusions

$$\mathbf{M}(\mathbf{E}^{(0)},\ n) \hookrightarrow \mathbf{M}(\mathbf{E}^{(1)},\ n) \hookrightarrow \mathbf{M}(\mathbf{E}^{(2)},\ n) \hookrightarrow \ldots$$

We investigate $\mathbf{M}(\mathbf{E}^{(i)}, n)$, for $n = 0, 1, 2$.

Firstly look at $\mathbf{M}(\mathbf{E}^{(0)}, n)$, where the 0-skeleton $\mathbf{E}^{(0)}$ is

$$\mathbf{E}^{(0)} : \quad \cdots \longrightarrow R \longrightarrow R \longrightarrow R \quad \xrightarrow{f} \quad B$$

with the $d_i^n = s_j^n =$ identity homomorphisms.

For $n = 0$, there is an equality

$$\mathbf{M}(\mathbf{E}^{(0)},\, 0) = E_0^{(0)}/d_1(\mathrm{Ker}d_0) = R,$$

and so $\mathbf{M}(\mathbf{E}^{(0)},\, 0)$ is just an algebra.

For $n = 1$, $\mathbf{M}(\mathbf{E}^{(0)}, 1)$ is

$$NE_1^{(0)}/\partial_2 NE_2^{(0)} \longrightarrow E_0.$$

It is easy to show that $NE_1^{(0)}/\partial_2 NE_2^{(0)}$ is trivial in the 0-skeleton $\mathbf{E}^{(0)}$ and hence

$$\mathbf{M}(\mathbf{E}^{(0)},\, 1) \cong (0 \longrightarrow R)$$

which is the crossed module by example 1.3.4.

And for $n = 2$, $\mathbf{M}(\mathbf{E}^{(0)},\, 2)$ is the trivial crossed square

$$\begin{array}{ccc} NE_2/d_3^3(NE_3) & \longrightarrow & \mathrm{Ker}d_0^1 \\ \downarrow & & \downarrow \\ \mathrm{Ker}d_1^1 & \longrightarrow & E_1 \end{array} \quad = \quad \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & R. \end{array}$$

Secondly take $\mathbf{M}(\mathbf{E}^{(1)}, n)$ and recall that the 1-skeleton $\mathbf{E}^{(1)}$ is

$$\mathbf{E}^{(1)} : \ldots R[s_0(\mathbf{S_2}), s_1(\mathbf{S_2})]) \underset{s_0, s_1}{\overset{d_0, d_1, d_2}{\rightrightarrows}} R[\mathbf{S_2}] \underset{s_0}{\overset{d_0, d_1}{\rightrightarrows}} R \xrightarrow{\ f\ } R/I.$$

For $n = 0$, it follows from section 4.3.2 that $\mathbf{M}(\mathbf{E}^{(1)},\ 0)$ is

$$E_0^{(1)}/d_1(\mathrm{Ker}d_0) \cong R/I$$

which is $\pi_0(\mathbf{E}^{(1)})$.

Let $n = 1$. Applying section 3.5 which implies the following result

$$\begin{array}{rcl} \mathbf{M}(\mathbf{E}^{(1)},\, 0) & = & (NE_1/\partial_2 NE_2 \longrightarrow E_0) \\ & = & (R^+[S_i]/P_1 \longrightarrow R) \end{array}$$

which is the free crossed module by theorem 1.4.2.

For $n=2$, $\mathbf{M}(\mathbf{E}^{(1)}, 2)$ simplifies to give (up to isomorphism)

$$\begin{array}{ccc} NE_2/d_3^3(NE_3) & \longrightarrow & \mathrm{Ker}d_0^1 \\ \downarrow & & \downarrow \\ \mathrm{Ker}d_1^1 & \longrightarrow & E_1 \end{array} \quad = \quad \begin{array}{ccc} R^+[s_0(\mathbf{S_2}),(\mathbf{S_2})]/P_2 & \longrightarrow & \overline{R^+[\mathbf{S_2}]} \\ \downarrow & & \downarrow \\ R^+[\mathbf{S_2}] & \longrightarrow & R[\mathbf{S_2}] \end{array}$$

which is a crossed square.

Let us next look for $\mathbf{M}(\mathbf{E}^{(2)}, n)$. Again recall the 2-skeleton $\mathbf{E}^{(2)}$

$$\ldots/(R[s_0(\mathbf{S_2}),s_1(\mathbf{S_2})])[\mathbf{S_3}] \underset{s_0,s_1}{\overset{d_0,d_1,d_2}{\rightrightarrows}} R[\mathbf{S_2}] \underset{s_0}{\overset{d_0,d_1}{\rightrightarrows}} R \xrightarrow{\ f\ } R/I.$$

The subsequent equalities can be easily obtained by direct calculation : for $n=0$,

$$\mathbf{M}(\mathbf{E}^{(2)}, 0) = E_0/d_1(\mathrm{Ker}d_0) \cong \pi_0(\mathbf{E}^{(2)}) = \mathbf{M}(\mathbf{E}^{(1)}, 0).$$

For $n=1$,

$$\mathbf{M}(\mathbf{E}^{(2)}, 1) \cong (R^+[S_i]/P_1 \to R) = \mathbf{M}(\mathbf{E}^{(1)}, 1).$$

Finally, let $n=2$. Since by an earlier result of this section, $\mathbf{M}(\mathbf{E}^{(2)},2)$ corresponds to the free crossed square, namely

$$\begin{array}{ccc} NE_2/d_3^3(NE_3) & \longrightarrow & \mathrm{Ker}d_0^1 \\ \downarrow & & \downarrow \\ \mathrm{Ker}d_1^1 & \longrightarrow & E_1 \end{array} \quad = \quad \begin{array}{ccc} (R[s_0(\mathbf{S_2}),(\mathbf{S_2})])^+[\mathbf{S_3}]/P_2 & \longrightarrow & \overline{R^+[\mathbf{S_2}]} \\ \downarrow & & \downarrow \\ R^+[\mathbf{S_2}] & \longrightarrow & R[\mathbf{S_2}]. \end{array}$$

Thus we have the following relations

$$\mathbf{M}(\mathbf{E}^{(2)}, 0) = \mathbf{M}(\mathbf{E}^{(1)}, 0), \qquad \mathbf{M}(\mathbf{E}^{(2)}, 1) = \mathbf{M}(\mathbf{E}^{(1)}, 1)$$

and

$$\mathbf{M}(\mathbf{E}^{(2)}, 2) = \mathbf{M}(\mathbf{E}^{(3)}, 2)$$

and so on. We present the following conjecture:

Conjecture

$$\left\{\mathbf{M}(\mathbf{E}^{(i)},\ n)\right\}_{i>1} \quad = \quad \mathbf{M}(\mathbf{E}^{(j)},\ n) \quad \text{if} \quad j \geq n+1.$$

5.5 Conclusions

Lower dimensional Peiffer elements for simplicial groups had been noted in [10]. In this thesis, we have extended these Peiffer elements for simplicial algebras to dimension four and given some technical results for higher dimensions. Up to dimension four, we have shown that

$$\partial_n(NE_n) = \sum_{\{I,J\}} K_I K_J$$

for $\emptyset \neq I, J \subset [n-1] = \{0, 1, \ldots, n-1\}$ with $I \cup J = [n-1]$, where

$$K_I = \bigcap_{i \in I} \mathrm{Ker} d_i \ \text{ and } \ K_J = \bigcap_{j \in J} \mathrm{Ker} d_j$$

by using the hand calculation. In general for $n > 4$, we can only prove

$$\sum_{\{I,J\}} K_I K_J \subseteq \partial_n(NE_n).$$

To prove the opposite inclusion, we have a general argument for $I \cap J = \emptyset$ and $I \cup J = [n-1]$. But for $I \cap J \neq \emptyset$, we could not say anything about it. One should be able to develop this result by means of the computer algebra software such as AXIOM or MAPLE.

Given the importance of the vanishing of these elements in the construction of the cotangent complex of Lichtenbaum and Schlessinger, [31], and the simplicial version of the cotangent complex of Quillen [39], André [1] and Illusie [26], it is natural to hope for higher order analogues of this result and for an analysis and interpretation of the structure of the resulting elements in NE_n, $n \geq 2$.

The free crossed modules for commutative algebras has been shown in [36] to be closely related to Koszul complexes (which is placed the last section of chapter one) that if (C, R, ∂) is a free crossed module R-module on a function $f : Y \to R$, with $Y = \{y_1, \ldots y_n\}$, then there is a natural isomorphism

$$C \cong R^n / \mathrm{Im} d,$$

where $d : \Lambda^2 R^n \to R^n$ is the Koszul differential. We believe that the material mentioned above together with section 3.4, may allow one to find if there is a deeper connection with the Koszul complex.

We have explored the relation between 2-Crossed Modules and Crossed Squares in chapter four and five. One ought to be able to examine the connections between 3-Crossed Modules, defined by Carrasco [12], and Crossed 3-cubes by applying the fourth order Peiffer elements. The freeness property of these structures can be obtained in terms of the 'step-by-step' construction with its 3-skeleton of a free simplicial algebras.

It is reasonable to expect all these results can be done for the other algebraic versions such as groups, groupoids, Lie algebras and so on.

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